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On the Theory of Dispersion Relations for Photon-Nucleon Scattering.

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Summary. — An exact proof of dispersion relations for Compton scattering on nucleons is given. The derivation is obtained following a method suggested by BOGOLJUBOV; only strong interactions are taken into account.

The method of dispersion relations is now widely used in quantum field theory. Although this method doesn't give complete information about specific processes, nevertheless it leads to some exact relations between the observed values, which could be directly compared with the experiment.

Dispersion relations attracted special interest since using them one can solve experimentally the problems of locality and causality in the elementary particle interaction. But to reach this aim it was necessary to prove exactly that dispersion relations have to be always valid when the mentioned properties are present.

However, in spite of the fact that the dispersion relation method was suggested in a number of papers as early as 1954-1955 [1], until recently the proof of dispersion relations was rather obscure.

(1) M. GELL-MANN, M. GOLDBERGER and W. THIRRING: *Phys. Rev.*, **95**, 1612 (1954). M. GOLDBERGER: *Phys. Rev.*, **97**, 508 (1955); H. KARPLUS and H. RUDERMAN: *Phys. Rev.*, **98**, 771 (1955).

It was only at the end of 1956 that N. N. BOGOLJUBOV (2), having overcome a number of difficulties, succeded in giving a satisfactory demonstration of dispersion relations for meson-nucleon scattering processes.

To achieve this result N. N. BOGOLJUBOV had to make a thorough analysis of the physical premises of the derivation and had to prove a number of complicated mathematical theorems on the behaviour at the boundary of functions of complex variables and on functional analysis.

The proof for the forward scattering was also obtained by H. LEHMANN, K. SYMANZIK and R. JOST (3).

The proof of dispersion relations for forward scattering (in absence of unobserved region) is rather simple and is based on the locality of meson or nucleon fields.

For an arbitrary scattering angle a deeper analysis is required, based on the locality of both meson and nucleon fields.

Dispersion relations for the Compton effect on nucleons were obtained in (1,4). The proof of these relations in absence of unobserved region was discussed in (5). In the present paper following the method suggested by BOGOLJUBOV (2,6) we shall give the exact proof that these relations hold true for an arbitrary scattering angle. As in (1,4) we shall not take into account here the electromagnetic corrections, restricting ourselves to the main term of the amplitude which is proportional to e^2, i.e. considering only strong interactions.

1. – The amplitude for Compton effect reaction.

Let us denote the photon annihilation operator by $a_\nu^{(-)}(q')$ and the production operator by $a_\nu^{(+)}(q)$. Then the matrix element for the Compton effect can be written as follows:

$$S(\alpha, q; \omega, q') = (2\pi)^3 \langle p\vec{s}' \,|\, a_{\nu'}^{(-)}(q') S a_\nu^{(+)}(q) \,|\, ps \rangle \,. \tag{1.1}$$

α and ω are initial and final states indices respectively. They involve all the quantum numbers describing the system except photon momenta q and q'.

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(2) N. N. BOGOLJUBOV: *Report at the International Conference on Theoretical Physics*, Seattle (September 1956); N. N. BOGOLJUBOV, B. V. MEDVEDJEV and M. K. POLIVANOV: *Problems Related to the Theory of Dispersion Relations* (in print).

(3) K. SYMANZIK: *Phys. Rev.*, **105**, 743 (1957); H. LEHMANN and R. JOST: *Report at the International Conference on Theoretical Physics*, Seattle (September 1956).

(4) N. N. BOGOLJUBOV and D. V. SHIRKOV: *Dokl. Acad. Sci.*, **113**, 527 (1957).

(5) A. A. LOGUNOV and A. FRENKIN: *Nuclear Physics* (in print).

(6) N. N. BOGOLJUBOV and D. V. SHIRKOV: *Introduction into the Theory of Quantum Fields* (1957), chapter IX.

It is easy to show that

$$(1.2)\qquad \begin{cases} [a_{\nu'}^{(-)}(q'),\, S] = \dfrac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2q_0'}} \int \exp\,[iq'x]\, \dfrac{\delta S}{\delta A_{\mu'}(x)}\, \mathrm{d}x l_{\mu'}^{\nu'}\,, \\ [S,\, a_{\nu}^{(+)}(q)] = \dfrac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2q_0}} \int \exp\,[-iqy]\, \dfrac{\delta S}{\delta A_{\mu}(y)}\, \mathrm{d}y\, l_{\mu}^{\nu}\,. \end{cases}$$

Commuting the production operator in (1.1) through to the left (we assume $q \neq q'$) and considering that

$$(1.3)\qquad S^{+}|ps\rangle = |ps\rangle\,,$$

we get

$$(1.4)\qquad S(\alpha, q; \omega, q') = -\,i\,\frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2q_0}} \int \exp\,[-iqy] \langle p's'\,|\, a_{\nu'}^{(-)}(q') j_{\mu}(y)\,|ps\rangle \mathrm{d}y l_{\mu}^{\nu}\,,$$

where

$$j_{\mu}(y) = i\,\frac{\delta S}{\delta A_{\mu}(y)}\, S^{+}\,,$$

Commuting the annihilation operator in (1.4) through to the right one obtains

$$(1.5)\qquad S(\alpha, q; \omega, q') = -\,i(4q_0q_0')^{-\frac{1}{2}} \sum_{\mu,\mu'} l_{\mu}^{\nu}\, l_{\mu'}^{\nu'} \cdot \int \mathrm{d}x\, \mathrm{d}y\, \exp\,[i(q'x - qy)] \langle p's'\,|\, \frac{\delta j_{\mu}(y)}{\delta A_{\mu'}(x)}\,|ps\rangle\,.$$

The amplitude $S(\alpha, q; \omega, q')$ is a retarded one, since

$$(1.6)\qquad \frac{\delta j_{\mu}(y)}{\delta A_{\mu'}(x)} = 0\,. \qquad x \lesssim y\,.$$

Using translation invariance the expression (1.5) may be written in the form:

$$(1.7)\qquad S(\alpha, q; \omega, q') = \frac{i(2\pi)^4}{\sqrt{4q_0q_0'}}\, \delta(p' + q' - p - q) \sum_{\mu,\mu'} l_{\mu}^{\nu}\, l_{\mu'}^{\nu'}\, T_{\alpha\omega}^{\mathrm{ret}}\left(\frac{q+q'}{2}\right),$$

where

$$(1.8)\qquad \begin{cases} T_{\alpha\omega}^{\mathrm{ret}}\left(\dfrac{q+q'}{2}\right) = -\int \exp\left[\dfrac{i}{2}(q+q')x\right] F_{\alpha\omega}^{\mathrm{ret}}(x)\,\mathrm{d}x\,, \\ F_{\alpha\omega}^{\mathrm{ret}}(x-y) = \exp\left[\left\{-\,i\left(\dfrac{p'-p}{2}\right)(x+y)\right\} \langle p's'\,|\, \dfrac{\delta j_{\mu}(y)}{\delta A_{\mu'}(x)}\,|ps\rangle\right]. \end{cases}$$

Let us introduce also the «advanced» amplitude $T^{\text{adv}}_{\alpha\omega}$:

$$(1.9)\qquad \begin{cases} T^{\text{adv}}_{\alpha\omega}\left(\dfrac{q+q'}{2}\right) = -\int \exp\left[\dfrac{i}{2}(q+q')x\right] F^{\text{adv}}_{\alpha\omega}(x)\,\mathrm{d}x\,, \\ F^{\text{adv}}_{\alpha\omega}(x-y) = \exp\left[\left\{-i\left(\dfrac{p'-p}{2}\right)(x+y)\right\} \langle p's' \,|\, \dfrac{\delta j_{\mu'}(x)}{\delta A_{\mu}(y)} \,|\, ps\rangle\right]. \end{cases}$$

Taking into account (1.6) which expresses the causality conditions, one can easily see that

$$(1.10)\qquad \begin{cases} F^{\text{ret}}_{\alpha\omega}(x) = 0\,, & x \lesssim 0\,, \\ F^{\text{adv}}_{\alpha\omega}(x) = 0\,, & x \gtrsim 0\,. \end{cases}$$

Since

$$(1.11)\qquad \frac{\delta j_{\mu}(y)}{\delta A_{\mu'}(x)} - \frac{\delta j_{\mu'}(x)}{\delta A_{\mu}(y)} = -i[j_{\mu'}(x) j_{\mu}(y) - j_{\mu}(y) j_{\mu'}(x)]\,,$$

then

$$(1.12)\qquad F^{\text{ret}}_{\alpha\omega}(x) - F^{\text{adv}}_{\alpha\omega}(x) = -i[F^{-}_{\alpha\omega}(x) - P_{\mu\mu'} F^{-}_{\alpha\omega}(-x)]\,,$$

where

$$(1.13)\qquad F^{-}_{\alpha\omega}(x-y) = \exp\left[\left\{-i\left(\frac{p'-p}{2}\right)(x+y)\right\} \langle p's' \,|\, j_{\mu'}(x) j_{\mu}(y) \,|\, ps\rangle\right].$$

Using a complete set of state vectors for the pion nucleon system, one can rewrite (1.13) in the form:

$$(1.14)\qquad F^{-}_{\alpha\omega}(x) = \frac{1}{(2\pi)^3} \sum_n \int \mathrm{d}\boldsymbol{k}_n \langle p's' \,|\, j_{\mu'}(0) \,|\, n, \boldsymbol{k}_n\rangle \langle n, \boldsymbol{k}_n \,|\, j_{\mu}(0) \,|\, ps\rangle \cdot \exp\left[\left\{-i\boldsymbol{k}_n x + \frac{i}{2}(p+p')x\right\}\right].$$

2. – Analytical properties of the Compton effect amplitude for virtual photons $q^2 = q'^2 = \tau < -\boldsymbol{p}^2$.

In order to study the analytical properties of the Compton effect amplitude let us choose the special co-ordinate system in which the sum of nucleon momenta before and after the scattering is equal to zero:

$$(2.1)\qquad \boldsymbol{p} + \boldsymbol{p}' = 0\,.$$

Using conservation laws of energy and momentum we have:

$$(2.2)\qquad \begin{cases} \boldsymbol{q} = \lambda \boldsymbol{e} - \boldsymbol{p}\,, \qquad \boldsymbol{e} \perp \boldsymbol{p}\,, \qquad |\boldsymbol{e}| = 1, \\ \boldsymbol{q}' = \lambda \boldsymbol{e} + \boldsymbol{p}\,, \quad q_0^2 = q_0'^2 = E^2 = \lambda^2 + \boldsymbol{p}^2 + \tau\,. \end{cases}$$

Suppose that $\tau < -\boldsymbol{p}^2$, then $\lambda^2 > E^2$. Using the variables E and τ the functions $T^{\mathrm{ret}}_{\alpha\omega}$, $T^{\mathrm{adv}}_{\alpha\omega}$ may be written:

$$(2.3)\qquad \begin{cases} T^{\mathrm{ret}}_{\alpha\omega}(E,\tau) = -\int \exp\,[i(Ex_0 - \sqrt{E^2 - \boldsymbol{p}^2 - \tau}\,\boldsymbol{e}\boldsymbol{x})] F^{\mathrm{ret}}_{\alpha\omega}(x)\,\mathrm{d}x\,, \\ T^{\mathrm{adv}}_{\alpha\omega}(E,\tau) = -\int \exp\,[i(Ex_0 - \sqrt{E^2 - \boldsymbol{p}^2 - \tau}\,\boldsymbol{e}\boldsymbol{x})] F^{\mathrm{adv}}_{\alpha\omega}(x)\,\mathrm{d}x\,. \end{cases}$$

Let us introduce the function $T_{\alpha\omega}(E,\tau)$ in the following way:

$$T_{\alpha\omega}(E,\tau) = T^{\mathrm{ret}}_{\alpha\omega}(E,\tau) - T^{\mathrm{adv}}_{\alpha\omega}(E,\tau)\,.$$

Taking into account (2.3), $T_{\alpha\omega}(E,\tau)$ becomes

$$(2.4)\qquad T_{\alpha\omega}(E,\tau) = \int \exp\,[i(Ex_0 - \sqrt{E^2 - \boldsymbol{p}^2 - \tau}\,\boldsymbol{e}\boldsymbol{x})] F_{\alpha\omega}(x)\,\mathrm{d}x\,,$$

where

$$F_{\alpha\omega}(x-y) = i \exp\left[-i\left(\frac{p'-p}{2}\right)(x+y)\right] \langle p's' | j_{\mu'}(x) j_{\mu}(y) - j_{\mu}(y) j_{\mu'}(x) | ps \rangle\,.$$

The functions T^{ret} and T^{adv} are here determined as generalized functions of E and τ only for real values of E and τ. However, they may be extended to complex values of E. Let us assume

$$(2.5)\qquad -V < \tau < -\boldsymbol{p}^2\,,$$

where V is an arbitrary but fixed positive number. Then

$$|\operatorname{Im} E| > |\operatorname{Im} \sqrt{E^2 - \boldsymbol{p}^2 - \tau}\,|\,,$$

for any E with $\operatorname{Im} E \neq 0$. But since $x_0 > |\boldsymbol{x}|$ (1.10), then if condition (2.5) is satisfied the functions ST^{ret} and ST^{adv} (*) are analytical functions

(*) In order to eliminate ambiguity of the square root $\sqrt{E^2 - \boldsymbol{p}^2 - \tau}$ it is necessary to deal with the symmetrical combination $\frac{1}{2}(T(\boldsymbol{e}) + T(-\boldsymbol{e}))$ and the antisymmetrical one $1/(2\lambda)(T(\boldsymbol{e}) - T(-\boldsymbol{e}))$ in $\boldsymbol{e}$. The symbol S implies symmetrization or antisymmetrization.

of E in the region $\operatorname{Im} E > 0$ and $\operatorname{Im} E < 0$ respectively. In order to decide whether these functions can be considered as a single analytical function regular for all $\operatorname{Im} E \neq 0$ it is necessary to investigate the behaviour of difference $\delta T(E, \tau)$ for real E. Substituting (1.14) into (2.4) and integrating over x and then over $\boldsymbol{k}_n$ we get

$$
\begin{aligned}
(2.6)\qquad \delta T_{\alpha\omega}(E,\tau) = {} & + 2\pi i \sum_n \delta\langle -\boldsymbol{p}s' | j_{\nu'}(0) | n, \lambda\boldsymbol{e}\rangle\langle n, \lambda\boldsymbol{e} | j_{\nu}(0) | ps\rangle \cdot \\
& \cdot \delta(E - \sqrt{\lambda^2 + M_n^2} + \sqrt{\boldsymbol{p}^2 + M^2}) + \\
& - 2\pi i \sum_n \delta\langle -\boldsymbol{p}, s' | j_{\nu}(0) | n, -\lambda\boldsymbol{e}\rangle\langle n, -\lambda\boldsymbol{e} | j_{\nu'}(0) | ps\rangle \cdot \\
& \cdot \delta(E + \sqrt{\lambda^2 + M_n^2} - \sqrt{\boldsymbol{p}^2 + M^2}) .
\end{aligned}
$$

Separating the term with intermediate one-nucleon state and making the elementary transformation of δ-functions we may write the expression (2.6) as follows:

$$
\begin{aligned}
(2.7)\qquad \delta T_{\alpha\omega}(E,\tau) = {} & \delta \cdot f_{\alpha\omega}(E,\tau) - 2\pi i \delta A(\boldsymbol{p},\tau)\, \delta(E + E_p) - \\
& - 2\pi i \delta B(\boldsymbol{p},\tau)\, \delta(E - E_p) ,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta f_{\alpha\omega}(E,\tau) = {} & + \pi i \sum_{n>1} \left| \frac{M_n^2 + M^2 - \tau}{M^2 + \boldsymbol{p}^2} \right| \delta(E + E_{pn}(\tau)) \cdot \\
& \cdot \delta\langle -\boldsymbol{p}'s' | j_{\nu'}(0) | n, \lambda\boldsymbol{e}\rangle\langle n, \lambda\boldsymbol{e} | j_{\nu}(0) | ps\rangle + \\
& - \pi i \sum_{n>1} \left| \frac{M_n^2 + M^2 - \tau}{M^2 + \boldsymbol{p}^2} \right| \cdot \delta(E - E_{pn}(\tau)) \delta\langle -\boldsymbol{p}s | j_{\nu}(0) | n, -\lambda\boldsymbol{e}\rangle\langle n, -\lambda\boldsymbol{e} | j_{\nu'}(0) | ps\rangle ,
\end{aligned}
$$

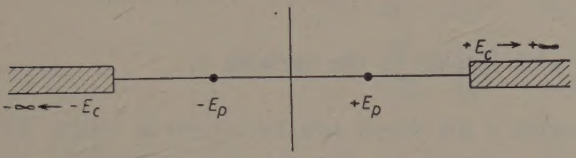
$$
\delta A(\boldsymbol{p},\tau) = -\left| \frac{M^2 - \tau/2}{M^2 + \boldsymbol{p}^2} \right| \sum_{s''} \delta\langle -\boldsymbol{p}s' | j_{\nu'}(0) s'', \lambda\boldsymbol{e}\rangle\langle s'', \lambda\boldsymbol{e} | j_{\nu}(0) | \boldsymbol{p}s\rangle ,
$$

$$
(2.8)\qquad \begin{cases}
\delta B(\boldsymbol{p},\tau) = \left| \dfrac{M^2 - (\tau/2)}{M^2 + \boldsymbol{p}^2} \right| \displaystyle\sum_{s''} \delta\langle -\boldsymbol{p}s' | j_{\nu}(0) | s'', -\lambda\boldsymbol{e}\rangle\langle -\lambda\boldsymbol{e}, s'' | j_{\nu'}(0) | \boldsymbol{p}s\rangle , \\
E_{p,n}(\tau) = \dfrac{\boldsymbol{p}^2 + (\tau/2) - (M_n^2/2) + (M^2/2)}{\sqrt{M^2 + \boldsymbol{p}^2}} , \\
E_p = \dfrac{\boldsymbol{p}^2 + (\tau/2)}{\sqrt{M^2 + \boldsymbol{p}^2}} .
\end{cases}
$$

If we suppose that the nucleon + π-meson system has no bound states with masses less than the sum of the nucleon and π-meson masses, i.e.:

$$
(2.9)\qquad M_n \geqslant M + \mu ,
$$

then for the momenta of the scattering $p^2 < M\mu + (\mu^2/2) - (\tau/2)$ the function $T_{\alpha\omega}(E, \tau)$ has the following « spectrum » in E (see Fig. 1). The continuous



$$E_c = \frac{M\mu - p^2 + (\mu^2/2) - (\tau/2)}{\sqrt{M^2 + p^2}},$$

Fig. 1.

spectrum is extended from $E = -\infty$ to $-E_c$ and then from E_c up to $E = +\infty$. For the values $E = \pm E_p$ the function $T_{\alpha\omega}(E, \tau)$ has δ-singularities. Since for the value of τ that we have chosen $\lambda^2 > 0$, the unobserved energy region is absolutely absent.

Thus, we have found that under the condition $-V < \tau < -p^2$ $ST^{\text{ret}}(E, \tau)$ *is an analytical function of* E *in the region* $\operatorname{Im} E > 0$ *and* $ST^{\text{adv}}(E, \tau)$ *is an analytical function of* E *in the region* $\operatorname{Im} E < 0$.

For real E the function $ST(E)$ is represented by the expression (2.7). If inequality (2.9) is verified there is a non-vanishing part of the real axis on which $ST(E, \tau) = 0$.

Thus we see, that ST^{ret} *and* ST^{adv} *may be considered as a single analytical function* $S\tilde{T}(E, \tau)$ *of* E *with cuts along the real axis for*

$$|E| > \frac{M\mu - p^2 + (\mu^2/2) - (\tau/2)}{\sqrt{M^2 + p^2}},$$

with the poles in the points $E = \pm E_p(\tau)$ *and increasing at infinity not faster than an* n^{th}*-power polynomial.*

Thus, for the function

$$g(E, \tau) = \frac{S\tilde{T}(E, \tau)}{(E - E_0)^{n+1}},$$

$$|E_0| < \frac{M\mu - p^2 + (\mu^2/2) - (\tau/2)}{\sqrt{M^2 + p^2}}, \qquad E_0 \neq E_p,$$

we can apply the Cauchy theorem with contour C (see Fig. 2)

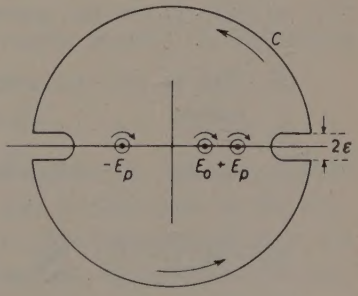


Fig. 2.

$$(2.10) \qquad S\tilde{T}(E, \tau) = \frac{(E - E_0)^{n+1}}{2\pi i} \int_{|E| \geq E_c}^{\infty} \frac{Sf(E', \tau)\, dE'}{(E' - E_0)^{n+1}(E' - E)} -$$

$$- \frac{SA(\tau, p)(E - E_0)^{n+1}}{(-E_p - E_0)^{n+1}(-E_p - E)} - \frac{SB(\tau, p)(E - E_0)^{n+1}}{(E_p - E_0)^{n+1}(E_p - E)} + \sum_{0 \leq r \leq n} C_r(\tau, p, E_0) E^r.$$

It must be noted that $\operatorname{Im} E \neq 0$ in (2.10).

Dispersion relations (2.10) are obtained here only for negative $-V < \tau < -\boldsymbol{p}^2$.

3. – Analytical continuation with respect to the variable τ.

In order to extend the region τ for which relation (2.10) is valid, let us apply the method of analytical continuation. For this it is necessary to study the analytical properties of the function $Sf(E, \tau)$. In the next paragraph it will be shown that if the variables E, τ, λ are real

$$-V \leqslant \tau < \varrho\mu^2 \tag{3.1}$$

(ϱ is some positive sufficiently small number and V is an arbitrary but fixed positive number) and

$$\boldsymbol{p}^2 < \sigma\mu^2 , \qquad \sigma = \left[\frac{2M + M}{M} + \frac{1}{4}\left(\frac{2M + \mu}{M + \mu}\right)^2\right],$$

then the function $Sf(E, \tau)$ may be represented in the form

$$Sf(E, \tau) = F_1(2E\sqrt{M^2 + \boldsymbol{p}^2} + \tau, \tau) + F_2(-2E\sqrt{M^2 + \boldsymbol{p}^2} + \tau, \tau) , \tag{3.2}$$

where $F_1(\xi, \tau)$ and $F_2(\xi, \tau)$ possess the following properties:

(3.3) *a*) $F_i(\xi, \tau)$ are the generalized functions of the variables ξ.

b) $F_i(\xi, \tau)$ are the analytical functions of the variable τ, regular in the region

$$-V < \operatorname{Re}\tau < \varrho\mu^2 ; \qquad |\operatorname{Im}\tau| < \varrho\mu^2 . \tag{3.4}$$

c) $F_i(\xi, \tau) = 0$ for $\xi < 2M\mu + \mu^2 - 2\boldsymbol{p}^2$.

Let us show now that from the representation (3.2) it follows directly that the dispersion relation (2.10) is true for the needed value

$$\tau = 0 . \tag{3.5}$$

Let us again take some negative τ satisfying the condition (2.5), then both dispersion relation (2.10) and representation (3.2) will be simultaneously valid for the given τ.

Substituting (3.2) into (2.10) we have

$$S\tilde{T}(E,\tau) = \Phi(E,\tau) + \left(\frac{E-E_0}{-E_p-E_0}\right)^{n+1}\cdot\frac{SA(\tau,\boldsymbol{p})}{E_p+E} + \left(\frac{E-E_0}{E_p-E_0}\right)^{n+1}\cdot\frac{SB(\tau,\boldsymbol{p})}{-E_p+E} + \sum_{0\le r\le n} C_r(\tau,\boldsymbol{p},E_0)E^r, \tag{3.6}$$

where

$$\Phi(E,\tau) = \frac{(E-E_0)^{n+1}}{2\pi i}\int_{\frac{2M\mu+\mu^2-2\boldsymbol{p}^2}{2\sqrt{M^2+\boldsymbol{p}^2}}}^{\infty}\frac{\mathrm{d}E''F_1(2E''\sqrt{M^2+\boldsymbol{p}^2},\tau)}{(E''-E-(\tau/2\sqrt{M^2+\boldsymbol{p}^2}))(E''-E_0-(\tau/2\sqrt{M^2+\boldsymbol{p}^2}))^{n+1}} + \tag{3.7}$$

$$+\frac{(E-E_0)^{n+1}}{2\pi i}\int_{\frac{2M\mu+\mu^2-2\boldsymbol{p}^2}{2\sqrt{M^2+\boldsymbol{p}^2}}}^{\infty}\frac{\mathrm{d}E''F_2(-2E''\sqrt{M^2+\boldsymbol{p}^2},\tau)}{(-E''-E+(\tau\,2\sqrt{M^2+\boldsymbol{p}^2}))(-E''-E_0+(\tau/2\sqrt{M^2+\boldsymbol{p}^2}))^{n+1}}.$$

The function $\Phi(E,\tau)$ is an analytical function of E in the region where none of the denominators

$$\left(E''-E-\frac{\tau}{2\sqrt{M^2+\boldsymbol{p}^2}}\right) \quad \text{and} \quad \left(-E''-E+\frac{\tau}{2\sqrt{M^2+\boldsymbol{p}^2}}\right),$$

become zero, i.e. if the following condition is verified

$$\mathrm{Im}\left(\frac{\tau}{2\sqrt{M^2+\boldsymbol{p}^2}}\pm E\right)\neq 0\,.$$

This will be always true if E and τ vary in a region limited by the strong inequality:

$$|\mathrm{Im}\,\tau| < 2\sqrt{M^2+\boldsymbol{p}^2}\,|\mathrm{Im}\,E|\,. \tag{3.8}$$

This is the only condition we must require in order to get the function $\Phi(E,\tau)$ to be analytical with respect to the variable E. Let us now pass to the analytical properties with respect to the variable τ. The functions $F_i(\xi,\tau)$ are analytical functions of τ in the region (3.4). Therefore, in order to get analiticity of the function $\Phi(E,\tau)$ in τ we must only require that the factors

$$\left(E'-E_0-\frac{\tau}{2\sqrt{M^2+\boldsymbol{p}^2}}\right)^{n+1}, \quad \left(-E'-E_0+\frac{\tau}{2\sqrt{M^2+\boldsymbol{p}^2}}\right)^{n+1},$$

never become zero.

It is quite obvious that if we choose

$$|E_0| < \frac{2M\mu+\mu^2-2\boldsymbol{p}^2-\tau}{2\sqrt{M^2+\boldsymbol{p}^2}} = E_c(\tau) \tag{3.9}$$

where τ is real, then this requirement is fulfilled. But our τ and p^2 ar erestricted by the inequalities $\tau < \varrho\mu^2$ for the real τ and $p^2 < \sigma\mu^2$. Therefore

$$|E_0| < \frac{2M\mu - 2\sigma\mu^2 - \varrho\mu^2 + \mu^2}{2\sqrt{M^2 + p^2}}. \tag{3.10}$$

Thus, the integrals (3.7) define the function $\Phi(E, \tau)$, analytical in the region

$$\begin{cases} -V \leqslant \operatorname{Re}\tau < \varrho\mu^2; & |\operatorname{Im}\tau| < \varrho\mu^2; \\ |\operatorname{Im}\tau| < 2\sqrt{M^2 + p^2}\,|\operatorname{Im} E|, \end{cases} \tag{3.11}$$

if

$$|E_0| < \frac{2M\mu - 2\sigma\mu^2 - \varrho\mu^2 + \mu^2}{2\sqrt{M^2 + p^2}}.$$

But on the other hand we stated earlier that the function $S\tilde{T}(E, \tau)$ is regular in the regions

$$\begin{cases} \operatorname{Im} E > |\operatorname{Im}\sqrt{E^2 - p^2 - \tau}|, \\ \operatorname{Im} E < -|\operatorname{Im}\sqrt{E^2 - p^2 - \tau}|. \end{cases} \tag{3.12}$$

Therefore, the difference

$$\{S\tilde{T}(E, \tau) - \Phi(E, \tau)\}(E^2 - E_p^2(\tau)), \tag{3.13}$$

must be an analytical function of the variables E, τ in the general part of the region (3.11) and (3.12), i.e. in the region

$$\begin{cases} -V \leqslant \operatorname{Re}\tau < \varrho\mu^2, & |\operatorname{Im}\tau| < 2\sqrt{M^2 + p^2}\,|\operatorname{Im} E|, \\ |\operatorname{Im}\tau| < \varrho\mu^2, & |\operatorname{Im}\sqrt{E^2 - p^2 - \tau}| < |\operatorname{Im} E|. \end{cases} \tag{3.14}$$

But for the real $\tau < -p^2$ according to the dispersion relation (2.10) this difference is the polynomial in E

$$\frac{(E - E_0)^{n+1}}{(-E_p - E_0)}(E - E_p(\tau))SA(\tau, p) + \left(\frac{E - E_0}{E_p - E_0}\right)^{n+1}(E + E_p(\tau))SB(\tau, p) + \tag{3.15}$$

$$+ (E^2 - E_p^2(\tau)) \sum_{0 \leq r \leq n} C_r(\tau, p, E_0)E^r.$$

Owing to the fact that the analytical continuation is unique, it must be polinomial in E in all the region (3.14). Therefore the functions $A(\tau, p)$, $B(\tau, p)$, $C_r(\tau, p, E_0)$ must allow analytical continuation to some region of complex values of τ.

Let us show that in any case this region involves the whole region

$$-V \leqslant \operatorname{Re}\tau < \varrho\mu^2\,, \qquad |\operatorname{Im}\tau| < \varrho\mu^2\,. \tag{3.16}$$

Indeed, let us choose some $\tau = \tau^*$, $\operatorname{Im}\tau^* \neq 0$ lying in the region (3.16) and show that it is always possible to choose for it a corresponding $E = E^*$ in such a way that the pair τ^*, E^* lies in the region (3.14). In fact, let us choose τ^* and E^* so that

$$\operatorname{Im}\tau^* = 2\operatorname{Re}E^*\cdot\operatorname{Im}E^*\,, \qquad \operatorname{Re}E^* < M\,, \qquad (\operatorname{Re}E^*)^2 - \operatorname{Re}\tau^* - \boldsymbol{p}^2 > 0\,. \tag{3.17}$$

Then

$$(\operatorname{Im}\sqrt{E^{*2} - \boldsymbol{p}^2 - \tau^*})^2 =$$

$$= \tfrac{1}{2}\{\sqrt{[\operatorname{Re}(E^{*2} - \boldsymbol{p}^2 - \tau^*)]^2 + [\operatorname{Im}(E^{*2} - \tau^*)]^2} + \operatorname{Re}\tau^* + \boldsymbol{p}^2 - \operatorname{Re}E^{*2}\}\,,$$

$$E^* = \operatorname{Re}E^* + i\operatorname{Im}E^*\,, \qquad E^{*2} = (\operatorname{Re}E^*)^2 - (\operatorname{Im}E^*)^2 + 2i\operatorname{Re}E^*\cdot\operatorname{Im}E^*\,,$$

$$\operatorname{Im}E^{*2} = 2\operatorname{Re}E^*\cdot\operatorname{Im}E^*\,, \qquad \operatorname{Im}(E^{*2} - \tau) = 0\,,$$

$$\operatorname{Im}\sqrt{E^{*2} - \boldsymbol{p}^2 - \tau^*} = 0\,, \qquad (\operatorname{Im}E^*)^2 \neq 0\,.$$

Therefore

$$|\operatorname{Im}\sqrt{E^{*2} - \boldsymbol{p}^2 - \tau}| < |\operatorname{Im}E^*|\,,$$

$$|\operatorname{Im}\tau^*| = 2|\operatorname{Re}E^*|\cdot|\operatorname{Im}E^*| < 2M|\operatorname{Im}E^*| < 2\sqrt{M^2 + \boldsymbol{p}^2}\cdot|\operatorname{Im}E^*|\,.$$

Thus, for any point τ^* in the region (3.16) with $\operatorname{Im}\tau^* \neq 0$ it is always possible to find an E^* such that the pair τ^*, E^* lies in the region (3.14).

From the above considerations it follows that the functions $A(\tau, \boldsymbol{p})$, $B(\tau, \boldsymbol{p})$, $C_r(\tau, \boldsymbol{p}, E_0)$ are analytical in the whole region (3.16) with a possible cut along the real axis. It can be shown that this cut is absent. In fact, let us choose τ_r real, in the following way

$$-V < \tau_r < \varrho\mu^2$$

and suppose

$$\tau = \tau_\pm = \tau_r \pm i\eta \qquad \eta > 0,$$

$$E = E_\pm = E_r \pm \frac{i\eta}{2E_r}; \qquad E_r > 0\,.$$

Now if we require

$$\boldsymbol{p}^2 + \varrho\mu^2 < E_r^2 < M^2$$

then for small η, $\tau_\pm$ an E will satisfy all the inequalities (3.17), and therefore

the points $\tau_\pm$ and $E_\pm$ belong to the region (3.14). Therefore $\Phi(E_\pm, \tau_\pm)$ is an analytical function of both arguments. But the points τ with $\operatorname{Im}\tau = 0$ ($\operatorname{Im}\tau \neq 0$) belong to the region (3.14), therefore by continuity

$$\lim_{\eta\to 0} \Phi(E_\pm, \tau_\pm) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \Phi(E_r \pm i\varepsilon, \tau_r)\,. \tag{3.18}$$

Using (3.7) we can find that

$$\lim_{\eta\to 0} \{\Phi(E_+, \tau_+) - \Phi(E_-, \tau_-)\} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \{\Phi(E_r + i\varepsilon, \tau_r) - \Phi(E_r - i\varepsilon, \tau_r)\} = \tag{3.19}$$

$$= F_1(2E_r\sqrt{M^2 + \boldsymbol{p}^2} + \tau_r, \tau_r) + F_2(-2E_r\sqrt{M^2 + \boldsymbol{p}^2} + \tau_r, \tau_r) = Sf(E_r, \tau_r)\,,$$

or considering (2.7)

$$\lim_{\eta\to 0} [\Phi(E_+, \tau_+) - \Phi(E_-, \tau_-)](E_r^2 - E_p^2(\tau_r)) = ST(E_r, \tau_r)(E_r^2 - E_p^2(\tau_r))\,. \tag{3.20}$$

E_+, τ_+ and E_-, τ_- belong to the region (3.14), then for such pairs of points we can use expressions (2.3) for the functions $ST^{\mathrm{ret}}(E_+, \tau_+)$, $ST^{\mathrm{adv}}(E_-, \tau_-)$ which in this case are meaningful and as $\eta \to 0$ converge to the functions

$$ST^{\mathrm{ret}}(E_r, \tau_r)\,, \qquad ST^{\mathrm{adv}}(E_r, \tau_r)\,.$$

Thus we get the following limit relation

$$\lim_{\eta\to 0} \{S\tilde{T}(E_+, \tau_+) - S\tilde{T}(E_-, \tau_-)\} = \tag{3.21}$$

$$= \{ST^{\mathrm{ret}}(E_r, \tau_r) - ST^{\mathrm{adv}}(E_r, \tau_r)\} = ST(E_r, \tau_r)\,.$$

Multiplying (3.21) by $(E_r^2 - E_p^2(\tau_r))$ and combining it with (3.20), we get

$$(E_r^2 - E_p^2(\tau_r)) \lim_{\eta\to 0} \{S\tilde{T}(E_+, \tau_+) - \Phi(E_+, \tau_+)\} = \tag{3.22}$$

$$= (E_r^2 - E_p^2(\tau_r)) \lim_{\eta\to 0} \{S\tilde{T}(E_-, \tau_-) - \Phi(E_-, \tau_-)\}.$$

Since the pairs of points (E_+, τ_+) and (E_-, τ_-) lie in the region (3.14), then, as it was shown previously, the function which appears in both sides of the relation (3.22) coincides with the polynomial in E. Denoting this polynomial by $P_\tau(E)$ we can re-write relation (3.22) as follows:

$$\lim_{\eta\to 0} P_{\tau_+}(E_+) = \lim_{\eta\to 0} P_{\tau_-}(E_-) \tag{3.23}$$

or by continuity

$$\lim_{\eta\to 0} P_{\tau_+}(E_r) = \lim_{\eta\to 0} P_{\tau_-}(E_r)\,, \tag{3.24}$$

i.e. $P_\tau(E)$ is a continuous function of τ when τ crosses the real axis. Since $P_\tau(E)$ is an analytical function of τ above and below the real axis, it remains an analytical function also on the real axis, i.e. the cut is absent. Therefore, the functions $A(\tau, \boldsymbol{p})$, $B(\tau, \boldsymbol{p})$, $C(\tau, \boldsymbol{p}, E_0)$ are analytical functions of τ for the region

$$-V \leqslant \operatorname{Re}\tau < \varrho\mu^2\,, \qquad |\operatorname{Im}\tau| < \varrho\mu^2\,.$$

Using the results obtained above concerning the analiticity of the functions A,B,C we may state that in the dispersion relation (2.10) the last three terms are analytical functions of τ and E for the region

$$-V \leqslant \operatorname{Re}\tau < \varrho\mu^2\,, \qquad |\operatorname{Im}\tau| < \varrho\mu^2\,, \qquad \operatorname{Im} E \neq 0.$$

But as $\Phi(E, \tau)$ is analytic in the region (3.11) then we can conclude that all the right hand side of the dispersion relations (3.10) is an analytical function in the region (3.11). Therefore we can extend the region used for the determination of $S\tilde{T}(E, \tau)$ in such a way that it coincides with the right hand side throughout the whole region (3.11).

The extended function $S\tilde{T}(E, \tau)$ satisfies the ordinary relations with improper limits

$$\lim_{\varepsilon\to 0} S\tilde{T}(E \pm i\varepsilon, \tau) = ST^{\substack{\text{ret}\\\text{adv}}}(E, \tau). \tag{3.25}$$

In fact, owing to continuity we have

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} S\tilde{T}(E \pm i\varepsilon, \tau) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} ST(E \pm i\varepsilon, \tau \pm 2iE\varepsilon)\,, \qquad |\alpha| < \frac{\sqrt{\boldsymbol{M}^2 + \boldsymbol{p}^2}}{E}\,.$$

Adding imaginary parts in such a way that the points $E \pm i\varepsilon$, $\tau \pm 2i\alpha E\varepsilon$ lie in the region (3.14), we may use the integral formulae (2.3) and write

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} S\tilde{T}(E + i\varepsilon, \tau) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} S\tilde{T}(E + i\varepsilon, \tau + 2i\alpha E\varepsilon) =$$

$$= S\int \exp\left[-\varepsilon x_0 + \boldsymbol{x}\boldsymbol{e}\,\operatorname{Im}\sqrt{(E \pm i\varepsilon)^2 - \boldsymbol{p}^2 - (\tau \pm 2i\alpha E\varepsilon)}\right] \cdot$$

$$\cdot \exp\left[iEx_0 - i\boldsymbol{x}\boldsymbol{e}\,\operatorname{Re}\sqrt{(E \pm i\varepsilon)^2 - \boldsymbol{p}^2 - (\tau \pm 2i\alpha E\varepsilon)}\right].$$

Note that

$$|\operatorname{Im}\sqrt{E_+^2 - \tau_+ - \boldsymbol{p}^2}| < \varepsilon\,.$$

Therefore

$$\lim_{\varepsilon\to 0} S\tilde{T}(E + i\varepsilon, \tau + 2i\alpha E\varepsilon) = ST^{\text{ret}}(E, \tau).$$

Analogously we have

$$\lim_{\varepsilon \to +0} S\tilde{T}(E - i\varepsilon,\ \tau - 2i\alpha E\varepsilon) = ST^{\text{adv}}(E, \tau)\,.$$

Thus, we find that in the region (3.11) both dispersion relations (2.10) and extreme transitions (3.25) are valid. But the point $\tau = 0$ (real photons) which is needed is included into the region (3.11). Therefore

$$(3.26) \qquad ST(E, 0) =$$

$$\frac{(E - E_0)^{n+1}}{2\pi i} \int\limits_{|E| = E_c} \frac{F_1(2E'\sqrt{M^2 + \boldsymbol{p}^2}, 0) + F_2(-2E'\sqrt{M^2 + \boldsymbol{p}^2}, 0)}{(E' - E)(E' - E_0)^{n+1}}\, \mathrm{d}E' +$$

$$+ \left(\frac{E - E_0}{-E_0 - E_p}\right)^{n+1} \cdot \frac{SA(0, \boldsymbol{p})}{(E_p + E)} + \left(\frac{E - E_0}{E_p - E}\right)^{n+1} \cdot \frac{SB(0, \boldsymbol{p})}{E - E_p} + \sum_{0 \le r \le n} C_r(0, \boldsymbol{p}, E_0) E^r\,.$$

Note, that the expression

$$(3.27) \qquad F_1(2E\sqrt{M^2 + \boldsymbol{p}^2}, 0) + F_2(-2E\sqrt{M^2 + \boldsymbol{p}^2}, 0)\,,$$

which replaces the function $Sf(E, 0)$ coincides with it in the region $|E| > |\boldsymbol{p}|$ and is its analytical continuation into the region

$$(3.28) \qquad \frac{M\mu - \boldsymbol{p}^2 + \mu^2/2}{\sqrt{M^2 + \boldsymbol{p}^2}} < |E| < |\boldsymbol{p}|\,,$$

where the direct determination of $Sf(E, 0)$ by means of the the integral (2.4) has no meaning. In order to go over to real values of E in (3.26) suppose that $E = E_r + i\varepsilon$ and let ε go to zero. Then, considering the expressions (3.25) we get choosing the corresponding sign of

$$(3.29) \qquad ST^{\text{ret}}_{\text{adv}}(E, 0) = \pm \frac{1}{2} ST(E, 0) + \frac{(E - E_0)^{n+1}}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{Sf(E', 0)\, \mathrm{d}E'}{(E' - E_0)^{n+1}(E' - E)} +$$

$$+ \left(\frac{E - E_0}{-E_p - E_0}\right)^{n+1} \frac{SA(0, \boldsymbol{p})}{(E + E_p)} + \left(\frac{E - E_0}{E_p - E_0}\right)^{n+1} \frac{SB(0, \boldsymbol{p})}{(E - E_p)} + \sum_{0 \le r \le n} C_r(0, \boldsymbol{p}, E_0) E^r\,.$$

Let us define Hermitian and anti-Hermitian part of the amplitude

$$(3.30) \qquad \begin{cases} D_{\alpha\omega}(E) = \dfrac{1}{2}\left(T^{\text{ret}}_{\alpha\omega}(E) + T^{\text{adv}}_{\alpha\omega}(E)\right), \\ A_{\alpha\omega}(E) = \dfrac{1}{2i}\left(T^{\text{ret}}_{\alpha\omega}(E) - T^{\text{adv}}_{\alpha\omega}(E)\right). \end{cases}$$

Note that

$$D_{\alpha\omega}(E) = D^{+}_{\alpha\omega}(E)\,, \qquad A_{\alpha\omega}(E) = A^{+}_{\alpha\omega}(E)\,.$$

With the help of (3.29) and (3.30) we obtain the following dispersion relations which connect the Hermitian and anti-Hermitian parts of the amplitude

$$(3.31)\qquad SD_{\alpha\omega}(E) =$$

$$= \frac{(E-E_0)^{n+1}}{\pi}\, P \int\limits_{|E|>\frac{M\mu-p^2+\mu^2/2}{\sqrt{M^2+p^2}}} \frac{SA_{\alpha\omega}(E')\,\mathrm{d}E'}{(E'-E)(E'-E_0)^{n+1}} + \left(\frac{E-E_0}{-E_p-E_0}\right)^{n+1} \cdot \frac{SA(0,\boldsymbol{p})}{(E+E_p)} +$$

$$+ \left(\frac{E-E_0}{E_p-E_0}\right)^{n+1} \cdot \frac{SB(0,\boldsymbol{p})}{E-E_p} + \sum_{0\leq r\leq n} C_r(0,\boldsymbol{p},E_0)E^r\,.$$

In the above discussion we made no assumption on the high frequency dependence of the scattering amplitude. If we suppose that the Compton scattering amplitude at infinity has a pole of order not higher thant the first, then, using « crossing symmetry » we can easily show that the dispersion relations (3.31) coincide with the relations (16)–(17) of paper ([4]).

4. – Investigation of the analytical properties of $ST_{\nu\omega}(E)$.

It was shown in 1 that

$$(4.1)\qquad \begin{cases} T_{\alpha\omega}\left(\dfrac{q+q'}{2}\right) = \displaystyle\int \mathrm{d}x \exp\left[i\left(\frac{q+q'}{2}\right)\right] F_{\alpha\omega}(x), \\ \exp\left[i\left(\dfrac{p'-p}{2}\right)(x+y)\right] F_{\alpha\omega}(x-y) = i\langle p's'\,|\,j_{\nu'}(x)j_{\nu}(y) - j_{\nu}(y)j_{\nu'}(x)\,|\,ps\rangle\,. \end{cases}$$

We can re-write the expression (4.1) as follows:

$$(4.2)\qquad \delta(p'-p+p_3+p_4)T_{\alpha\omega}\left(\frac{p'-p}{2}+p_3\right) =$$

$$= \frac{i}{(2\pi)^4}\,\bar{U}(p')\cdot\int \mathrm{d}x_1\cdot\mathrm{d}x_2\cdot\mathrm{d}x_3\cdot\mathrm{d}x_4\cdot$$

$$\cdot D(x_1,x_2,x_3,x_4)\cdot\exp\left[\{i(p'x_1-px_2+p_3x_3+p_4x_4)\}\right]u(p)\,,$$

where

$$(4.3)\qquad D(x_1x_2x_3x_4) = \langle 0\,\Big|\,\frac{\delta^2}{\delta\bar{\psi}(x_1)\,\delta\psi(x_2)}\,[j_{\nu'}(x_3),\,j_{\nu}(x_4)]_-\,\Big|\,0\rangle\,.$$

Since $D(x_1, x_2, x_3, x_4)$ is a translation-invariant function, then

$$(4.4)\quad \begin{cases} \int D(x_1, x_2, x_3, x_4) \exp\left[i(p_1x_1 \dots + p_4x_4)\right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 = \\ \qquad\qquad = (2\pi)^4 \delta(p_1 + \dots p_4)\tilde{D}(p_1, + \dots p_4)\,, \\ D(x_1, x_2, x_3, x_4) = D(x_1 + a, x_2 + a, x_3 + a, x_4 + a)\,. \end{cases}$$

It is quite obvious that

$$(4.5)\quad T_{\alpha\omega}\left(\frac{q+q'}{2}\right) = \frac{i}{(2\pi)^4}\, \bar{U}_{s'}(p')\tilde{D}(p', -p, q', -q)u_s(p)\,.$$

Performing variational differentiation in (4.3) we find that

$$(4.6)\quad D(x_1, x_2, x_3, x_4) = \sum_{1\le a\le 4} (1 - P_{\nu\nu'}P_{31})D^a(x_1, x_2, x_3, x_4)$$

(P is an operator of index permutation) where

$$(4.7)\quad \begin{cases} D^{(1)}(x_1, x_2, x_3, x_4) = \langle 0 \left| \dfrac{\delta j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)} \cdot \dfrac{\delta j_\nu(x_4)}{\delta\psi(x_2)} \right| 0\rangle\,, \\ D^{(2)}(x_1, x_2, x_3, x_4) = -\langle 0 \left| \dfrac{\delta j_{\nu'}(x_3)}{\delta\psi(x_2)} \cdot \dfrac{\delta j_\nu(x_4)}{\delta\psi(x_1)} \right| 0\rangle\,, \\ D^{(3)}(x_1, x_2, x_3, x_4) = \langle 0 \left| \dfrac{\delta^2 j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)\delta\psi(x_2)} \cdot j_\nu(x_4) \right| 0\rangle\,, \\ D^{(4)}(x_1, x_2, x_3, x_4) = -\langle 0 \left| j_\nu(x_4) \cdot \dfrac{\delta^2 j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)\delta\psi(x_2)} \right| 0\rangle\,. \end{cases}$$

In complete analogy with the operation (4.6) we can perform decomposition of the function $\tilde{D}(P_1, P_2, P_3, P_4)$.

Let us study in greater detail the Fourier transforms of the functions $D^i(x_1, x_2, x_3, x_4)$. Consider the expression $D^{(3)}$:

$$D^{(3)}(x_1, x_2, x_3, x_4) = \langle 0 \left| \frac{\delta^2 j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)\delta\psi(x_3)} \right| 0\rangle\langle 0 \left| j_\nu(x_4) \right| 0\rangle + \\ + \sum_n \int \mathrm{d}\boldsymbol{k} \langle 0 \left| \frac{\delta^2 j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)\delta\psi(x_2)} \right| n, \boldsymbol{k}\rangle\langle n\boldsymbol{k} \left| j_\nu(x_4) \right| 0\rangle\,.$$

But since

$$\langle n, \boldsymbol{k} \left| j_\nu(x_4) \right| 0\rangle = \langle n, \boldsymbol{k} \left| j_\nu(0) \right| 0\rangle \exp\left[ik_n x_n\right]\,, \qquad \boldsymbol{k}_n = \boldsymbol{k}, \qquad k_n^0 = \sqrt{\boldsymbol{k}^2 + M_n^2}\,,$$

and $\langle n, \boldsymbol{k} \left| j_\nu(0) \right| 0\rangle = 0$ for $M_n < 2\mu$ then in the Fourier expansion $D^3(x_1 \dots x_4)$·

$\cdot \exp[iP_4x_4]$ exponentials are present only with $P_4^2 \geqslant (2\mu)^2$. Thus $\tilde{D}^3(P_1 \dots P_4)=0$ if $P_4^2 < (2\mu)^2$. Analogously we have

$$\tilde{D}^{(4)}(P_1 \dots P_4) = 0 \qquad \text{if } P_4^2 < (2\mu)^2,$$

$$P_{\nu\nu'}P_{34}\,\tilde{D}^{(3)}(P_1 \dots P_4) = 0 \qquad \text{if } P_3^2 < (2\mu)^2,$$

$$P_{\nu\nu'}P_{34}\,\tilde{D}^{(4)}(P_1 \dots P_4) = 0 \qquad \text{if } P_3^2 < (2\mu)^2,$$

Consequently

$$\tilde{D}(P_1, P_2, P_3, P_4) = \sum_{a=1.2} (1 - P_{\nu\nu'}P_{34})\,\tilde{D}^{(a)}(P_1 \dots P_4) \tag{4.8}$$

if

$$P_3^2 < (2\mu)^2\,, \qquad P_4^2 < (2\mu)^2\,.$$

Theorem (*). Let four groups $(i = r, a;\ j = r, a)$ of translation-invariant generalized functions $F_{ij}^{\nu}(x_1 \dots x_4)$ be given which are transformed with linear finite-dimension representations of the Lorentz group and which possess the properties

$$\begin{cases} F_{rr}^{(\nu)}(x_1 \dots x_4) = 0 & \text{if } x_1 \lesssim x_3 \text{ or } x_2 \lesssim x_4, \\ F_{ra}^{(\nu)}(x_1 \dots x_4) = 0 & \text{if } x_1 \lesssim x_3 \text{ or } x_4 \lesssim x_2, \\ F_{ar}^{(\nu)}(x_1 \dots x_4) = 0 & \text{if } x_3 \lesssim x_1 \text{ or } x_2 \lesssim x_4, \\ F_{aa}^{(\nu)}(x_1 \dots x_4) = 0 & \text{if } x_3 \lesssim x_1 \text{ or } x_4 \lesssim x_2. \end{cases} \tag{4.9}$$

Let, besides, the function satisfy the conditions

$$\begin{cases} \tilde{F}_{rj}^{(\nu)}(P_1 \dots P_4) - \tilde{F}_{aj}^{(\nu)}(P_1 \dots P_4) = 0\,, & \text{if } P_1^2 < (M+\mu)^2 \text{ and } P_3^2 < (2\mu)^2 \\ \tilde{F}_{ir}^{(\nu)}(P_1 \dots P_4) - \tilde{F}_{ia}^{(\nu)}(P_1 \dots P_4) = 0\,, & \text{if } P_2^2 < (M+\mu)^2 \text{ and } P_4^2 < (2\mu)^2 \end{cases} \tag{4.10}$$

and also the requirements

$$F_{ij}^{(\nu)}(P_1 \dots P_4) = 0\,, \quad \text{if } (P_1+P_3)^2 < (M+\mu)^2 \text{ or } P_{10}+P_{30} < 0. \tag{4.11}$$

Then we can construct the generalized functions $\Phi_\omega(z_1 \dots z_5;\ z_6)$ of the real variable z_6 which are analytical functions of the complex variables $z_1 \dots z_5$ with the following properties:

(*) The proof of this theorem is given in a preprint of a paper by V. S. VLADIMIROV and A. A. LOGUNOV, to appear in *Izv. Akad. Nauk SSSR* (1958).

(7) N. N. BOGOLJUBOV and V. S. VLADIMIROV: *On the Analytical Continuation of Generalized Functions*, preprint JINR (1957).

1). The functions Φ_ω are regular in the region

$$(4.12)\quad \begin{cases} |z_1 - M^2| \leqslant \varrho\mu^2\,, & |z_2 - M^2| \leqslant \varrho\mu^2\,, & |z_3 - \tau| \leqslant \varrho\mu^2\,, \\ |z_4 - \tau| \leqslant \varrho\mu^2\,, & |z_5 + \alpha^2| \leqslant \varrho\mu^2(M/t)^2\,, & -V \leqslant \tau \leqslant \varrho\mu^2\,, \\ 0 \leqslant \alpha^2 \leqslant 4\sigma\mu^2\,, & t \geqslant \tfrac{1}{2}(M+\mu)\,, \end{cases}$$

ϱ is a small enough positive number and τ is a real number;

2) The function $\Phi_\omega(z_1 \dots z_5; z_6) = 0$ if

$$(4.13)\qquad z_a < \tfrac{1}{4}(M+\mu)\,.$$

3) For real $P_1 \dots p_4$ such that $P_1 + P_2 + P_3 + P_4 = 0$, the variables

$$(4.14)\quad \begin{aligned} z_1 &= P_1^2\,, \quad z_2 = P_2^2\,, \quad z_3 = P_3^2; \quad z_4 = P_4^2\,, \\ z_5 &= (P_1+P_2)^2, \quad z_6 = \frac{1}{4}(P_1+P_3)^2\,, \end{aligned}$$

satisfy the inequality (1.20) and $z_6 = ((P_1+P_3)/2)^2$, the functions $F_{ij}^{(\nu)}$ can be represented in the form of a sum

$$(4.15)\qquad \widetilde{F}_{ij}^{(\nu)}(P_1 \dots P_4) = \sum_\omega P_{i_1}^{\alpha_1} \dots P_{i_s}^{\alpha_s} \Phi_\omega(z_1, z_2, \dots, z_5; z_6)$$

if $P_1^2 + P_3^2 > 0$ with a finite number of terms $\omega = \{\alpha_1 \dots \alpha_s\}$.

Let us check whether the functions $D^{(a)}(x_1 \dots x_4)$ $(a = 1, 2)$ satisfy the conditions of the theorem stated above. Let us consider the function

$$D^{(1)}(x_1 \dots x_4)\,.$$

Suppose that

$$(4.16)\qquad D^{(1)}(x_1 \dots x_4) = D_{rr}^{(1)}(x_1 \dots x_4) = \langle 0 \left| \frac{\delta j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)} \cdot \frac{\delta j_\nu(x_4)}{\delta\psi(x_2)} \right| 0 \rangle\,,$$

and introduce the functions $D_{ar}^{(1)}$, $D_{ra}^{(1)}$, $D_{aa}^{(1)}$ as follows:

$$(4.17)\quad \begin{cases} D_{ar}^{(1)}(x_1 \dots x_4) = \langle 0 \left| \dfrac{\delta\Lambda(x_1)}{\delta A_{\nu'}(x_3)} \cdot \dfrac{\delta j_\nu(x_4)}{\delta\psi(x_2)} \right| 0 \rangle\,, \\ D_{ra}^{(1)}(x_1 \dots x_4) = \langle 0 \left| \dfrac{\delta j_{\nu'}(x_3)}{\delta\bar{\psi}(x_1)} \cdot \dfrac{\delta\bar{\Lambda}(x_2)}{\delta A_\nu(x_4)} \right| 0 \rangle\,, \\ D_{aa}^{(1)}(x_1 \dots x_4) = \langle 0 \left| \dfrac{\delta\Lambda(x_1)}{\delta A_{\nu'}(x_3)} \cdot \dfrac{\delta\bar{\Lambda}(x_2)}{\delta A_\nu(x_4)} \right| 0 \rangle\,, \\ \Lambda(x) = i\dfrac{\delta S}{\delta\bar{\psi}(x)} S^+; \qquad \bar{\Lambda}(x) = i\dfrac{\delta S}{\delta\psi(x)} S^+\,. \end{cases}$$

The functions $D_{ij}^{(1)}(x_1 \dots x_4)$ are translation-invariant and satisfy (owing to causality conditions) the requirement of (4.9). One may prove that they satisfy (4.10) as well. In fact, using the obvious identity

$$(4.18)\qquad \begin{cases} \dfrac{\delta j_{\nu'}(x_3)}{\delta \bar{\psi}(x_1)} - \dfrac{\delta \Lambda(x_1)}{\delta A_{\nu'}(x_3)} = i[j_{\nu'}(x_3), \Lambda(x_1)]_- , \\ \dfrac{\delta j_{\nu}(x_4)}{\delta \psi(x_2)} - \dfrac{\delta \bar{\Lambda}(x_2)}{\delta A_{\nu}(x_4)} = i[j_{\nu}(x_4), \bar{\Lambda}(x_2)]_- , \end{cases}$$

we have

$$D_{rr}^{(1)}(x_1 \dots x_4) - D_{ar}^{(1)}(x_1 \dots x_4) = \langle 0 \left| \left(\frac{\delta j_{\nu'}(x_3)}{\delta \bar{\psi}(x_1)} - \frac{\delta \Lambda(x_1)}{\delta A_{\nu'}(x_3)} \right) \frac{\delta j_{\nu}(x_4)}{\delta \psi(x_2)} \right| 0 \rangle =$$

$$= i\langle 0 \left| j_{\nu'}(x_3) \Lambda(x_1) \frac{\delta j_{\nu}(x_4)}{\delta \psi(x_2)} \right| 0 \rangle - i\langle 0 \left| \Lambda(x_1) j_{\nu'}(x_3) \frac{\delta j_{\nu}(x_4)}{\delta \psi(x_2)} \right| 0 \rangle .$$

Thus it is obvious that the Fourier transform of the first term is equal to zero if $P_3^2 < (2\mu)^2$ and the Fourier transform of the second term is equal to zero if $P_1^2 < (M+\mu)^2$ since $\langle 0 | \Lambda(0) | n, \boldsymbol{k} \rangle = 0$, if $M_n < M + \mu$.

Therefore

$$\tilde{D}_{rr}^{(1)}(P_1 \dots P_4) - \tilde{D}_{ar}^{(1)}(P_1 \dots P_4) = 0 , \quad \text{if} \quad P_1^2 < (M+\mu)^2, \; P_3^2 < (2\mu)^2 .$$

And similarly

$$\tilde{D}_{rj}^{(1)}(P_1 \dots P_4) - \tilde{D}_{aj}^{(1)}(P_1 \dots P_4) = 0 , \quad \text{if} \quad P_1^2 < (M+\mu)^2 , \quad P_3^2 < (2\mu)^2,$$

$$\tilde{D}_{jr}^{(1)}(P_1 \dots P_4) - \tilde{D}_{ja}^{(1)}(P_1 \dots P_4) = 0 , \quad \text{if} \quad P_2^2 < (M+\mu)^2 , \quad P_4^2 < (2\mu)^2.$$

Expansion of $D_{ij}^{(1)}(x_1 \dots x_4)$ using a complete set of state-vectors shows that these functions do not satisfy (4.11) because a one nucleon state is present.

However, if instead of $D_{ij}^{(1)}(x_1 \dots x_4)$ we introduce the functions

$$\left\{ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right)^2 + M^2 \right\} D_{ij}^{(1)}(x_1 \dots x_4),$$

then we can show that their Fourier transforms satisfy the conditions of (4.11). It is easily checked, e.g. for $D_{rr}^{(1)}(x_1 \dots x_4)$. Indeed, complete state expansion now gives us:

$$\left\{ \left(\frac{\partial}{c\, x_1} + \frac{\partial}{\partial x_3} \right)^2 + M^2 \right\} D_{rr}^{(1)}(x_1 \dots x_4) =$$

$$= \sum_n \int \mathrm{d}\boldsymbol{k}_n \langle 0 \left| \left\{ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right)^2 + M^2 \right\} \cdot \left(\frac{\delta j_{\nu'}(x_3)}{\delta \bar{\psi}(x_1)} \right) \right| n, \boldsymbol{k} \rangle \langle n, \boldsymbol{k} \left| \frac{\delta j_{\nu}(x_4)}{\delta \bar{\psi}(x_2)} \right| 0 \rangle .$$

Nucleon charge conservation requires that the expression

$$\langle n, \boldsymbol{k} \left| \frac{\delta j_\nu(x_4)}{\delta \psi(x_2)} \right| 0 \rangle ,$$

is equal to zero for any state without nucleons. The one-nucleon state disappears owing to the action of the differential operator.

Therefore

$$\{(P_1+P_3)^2 - M^2\}\tilde{D}^{(1)}_{rr}(P_1 \dots P_4) = 0 , \qquad (P_1+P_3)^2 < (M+\mu)^2 \quad \text{or} \quad P_{10}+P_{30} < 0.$$

Thus, the functions $\{(P_1+P_3)^2 - M^2\}\tilde{D}^{(1)}_{ij}(P_1 \dots P_4)$ satisfy all the conditions of the theorem and, therefore, can be represented in the form

$$(4.19) \quad [M^2-(P_1+P_3)^2]\tilde{D}^{(1)}(P_1 \dots P_4) = \begin{cases} \sum\limits_{\omega} P^{\alpha_1}_{i_1} \dots P^{\alpha_s}_{i_s} \Phi_\omega(P_1^2 \dots P_4^2; (P_1+P_3)^2) \\ \text{if } (P_1+P_3)^2 > (M+\mu)^2 \text{ and } P_{10}+P_{30} > 0, \\ 0, \text{ if } (P_1+P_3)^2 < (M+\mu)^2 \text{ or } P_{10}+P_{30} < 0 \end{cases}$$

For the function $P_{\nu\nu'}P_{34}\tilde{D}^{(1)}(P_1 \dots P_4)$, $\tilde{D}^{(2)}(P_1 \dots P_4)$, $P_{\nu\nu'}P_{34}\tilde{D}^{(2)}$ we can get relations analogous to (4.19) if we repeat the discussion identical to that which was applied to the function $D^{(1)}(P_1 \dots P_4)$ Thus we have

$$(4.20) \quad [M^2-(P_1+P_4)^2]P_{\nu\nu'}P_{34}\tilde{D}^{(1)}(P_1 \dots P_4) = \begin{cases} \sum\limits_{\omega} P^{\alpha_1}_{i_1} \dots P^{\alpha_s}_{i_s} \Phi_\omega(P_1^2 \dots; (P_1+P_4)^2 \\ \text{if } (P_1+P_4)^2 > (M+\mu)^2 \text{ and } P_{10}+P_{40} > 0, \\ 0, \text{ if } (P_1+P_4)^2 < (M+\mu)^2 \text{ or } P_{10}+P_{40} < 0 \end{cases}$$

$$(4.21) \quad [M^2-(P_1+P_4)^2]\tilde{D}^{(2)}(P_1 \dots P_4) = \begin{cases} \sum\limits_{\omega} P^{\alpha_1}_{i_1} \dots P^{\alpha_s}_{i_s} \Phi_\omega(P_1^2 \dots; (P_1+P_4)^2) \\ \text{if } (P_1+P_4)^2 > (M+\mu)^2 \text{ and } P_1^0+P_4^0 < 0, \\ 0, \text{ if } (P_1+P_4)^2 < (M+\mu)^2 \text{ or } P_{10}+P_{40} > 0 \end{cases}$$

$$(4.22) \quad [M^2-(P_1+P_3)^2]P_{\nu\nu'}P_{34}\tilde{D}^{(2)}(P_1 \dots P_4) = \begin{cases} \sum\limits_{\omega} P^{\alpha_1}_{i_1} \dots P^{\alpha_s}_{i_s} \Phi_\omega(P_1^2 \dots; (P_1+P_3)^2) \\ \text{if } (P_1+P_3)^3 > (M+\mu)^2 \text{ and } P_1^0+P_3^0 < 0, \\ 0, \text{ if } (P_1+P_3) < (M+\mu)^2 \text{ or } P_1^0+P_3^0 > 0. \end{cases}$$

In the frame of reference $\boldsymbol{p}+\boldsymbol{p}'=0$ the inequalities that were in (4.19)–(4.22) will have the following form:

$$
\left\{\begin{array}{lll}
E \gtrless E_c & \text{instead of} & (P_1+P_3)^2 \gtrless (M+\mu)^2\,, \\
E \lesseqgtr -E_c & \text{»} & (P_1+P_4)^2 \gtrless (M+\mu)^2\,, \\
E \gtrless -\sqrt{M^2+\boldsymbol{p}^2} & \text{»} & P_1^0+P_3^0 \gtrless 0\,, \\
E \lesseqgtr \sqrt{M^2+\boldsymbol{p}^2} & \text{»} & P_1^0+P_4^0 \gtrless 0\,.
\end{array}\right. \tag{4.23}
$$

Since the theorem requires (see (4.12)) $\boldsymbol{p}^2<\sigma\mu^2$ then we have $E_c>0$ and therefore the inequalities

$$(P_1+P_3)^2>(M+\mu)^2\,, \qquad P_1^0+P_3^0<0\,,$$

or

$$(P_1+P_4)^2>(M+\mu)^2\,, \qquad P_1^0+P_4^0<0\,,$$

are incompatible. This means that for the momenta $\boldsymbol{p}^2<\sigma\mu^2$ the functions $\widetilde{D}^{(2)}$ and $P_{\nu\nu'}P_{34}\widetilde{D}^{(2)}$ according to (4.21) (4.22) don't give contribution to the Fourier transform of the function $\widetilde{D}(P_1 \dots P_4)$. Thus, for the function $\widetilde{D}(P_1 \dots P_4)$ we get the following representation:

$$
\left\{\begin{array}{l}
\widetilde{D}(P_1 \dots P_4) = -\dfrac{1}{2\sqrt{M^2+\boldsymbol{p}^2}}\cdot\dfrac{1}{E+E_p}\cdot \\
\cdot \sum\limits_{\omega} p_{i_1}^{\alpha_1}\dots p_{i_s}^{\alpha_s}\Phi_\omega(M^2, M^2, \tau, \tau, -4\boldsymbol{p}^2, \tau+2E\sqrt{M^2+\boldsymbol{p}^2}+M^2+2\boldsymbol{p}^2)+ \\
+\dfrac{1}{2\sqrt{M^2+\boldsymbol{p}^2}}\cdot\dfrac{1}{E_p-E}\sum\limits_{\omega} p_{i_1}^{\alpha_1}\dots p_{i_s}^{\alpha_s}\Phi'_\omega\cdot \\
\qquad\qquad \cdot(M^2, M^2, \tau, \tau, -4\boldsymbol{p}^2, \tau-2E\sqrt{M^2+\boldsymbol{p}^2}+M^2+2\boldsymbol{p}^2),
\end{array}\right. \tag{4.24}
$$

where

$$\Phi_\omega=0\,, \quad E<E_c\,; \qquad \Phi'_\omega=0\,, \quad E>-E_c\,.$$

Since in virtue of (2.7) in the region $|E|>E_c$ we have

$$ST(E,\tau)=S\cdot f(E,\tau)\,,$$

then with the following notations

$$
(4.25)\quad
\begin{cases}
F_1(\tau + 2E\sqrt{M^2 + \boldsymbol{p}^2}, \tau) = -\dfrac{i}{(2\pi)^4}\cdot \\
\cdot\, \bar{u}_{s'}(p')\dfrac{\sum\limits_{\omega} p_{i_1}^{\alpha_1} \dots p_{i_s}^{\alpha_s} \Phi_\omega(M^2, M^2, \tau, \tau, -4\boldsymbol{p}^2, \tau + 2E\sqrt{M^2 + \boldsymbol{p}^2})}{2\sqrt{M^2 + \boldsymbol{p}^2}(E + E_p)} u_s(p)\,, \\
F_2(\tau - 2E\sqrt{M^2 + \boldsymbol{p}^2}, \tau) = -\dfrac{i}{(2\pi)^4}\cdot \\
\cdot\, \bar{u}_{s'}(p')\dfrac{\sum\limits_{\omega} p_{i_1}^{\alpha_1} \dots p_{i_s}^{\alpha_s} \Phi_\omega(M^2, M^2, \tau, \tau, -4\boldsymbol{p}^2, \tau - 2E\sqrt{M^2 + \boldsymbol{p}^2})}{2\sqrt{M^2 + \boldsymbol{p}^2}(E - E_p)} u_s(p)\,,
\end{cases}
$$

and using (4.5) we easily get:

$$
(4.26)\qquad Sf(E, \tau) = F_1(2E\sqrt{M^2 + \boldsymbol{p}^2} + \tau, \tau) + F_2(-2E\sqrt{M^2 + \boldsymbol{p}^2} + \tau, \tau)\,.
$$

Since the functions $F_i(\xi, \tau)$ are combinations of the functions $\Phi_\omega(z_1 \dots z_5; z_6)$ then according (4.12)–(4.15) they have the following properties:

a) $F_i(\xi, \tau)$ are generalized functions of the variable ξ;

b) $F_i(\xi, \tau)$ are analytical functions of the variable τ, and regular in the region

$$
-V \leqslant \operatorname{Re}\tau < \varrho\mu^2\,, \qquad |\operatorname{Im}\tau| < \varrho\mu^2\,;
$$

c) $F_i(\xi, \tau) = 0$ for $\xi < 2M\mu + \mu^2 - 2\boldsymbol{p}^2$.

It must be noted that owing to (4.12) the representation (4.26) is valid, if $\boldsymbol{p}^2 < \sigma\mu^2$.

* * *

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APPENDIX

In order to calculate A and B in the dispersion relations (2.10) it is necessary to study the expression

$$
(A.1)\qquad \langle \boldsymbol{p}''s'' | j_\nu(0) | \boldsymbol{p}s \rangle\,.
$$

With the help of a formula analogous to (1.2) this expression can be written in the following form:

$$\text{(A.2)} \qquad \langle \boldsymbol{p}''s'' | j_\nu(0) | \boldsymbol{p}s \rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \bar{u}^{s''}(p'') \int \mathrm{d}x \exp [ip''x] \langle 0 \left| \frac{\delta j_\nu(0)}{\delta \bar{\psi}(x)} \right| \boldsymbol{p}s \rangle ,$$

$$p_0'' = \sqrt{\boldsymbol{p}''^2 + M^2} .$$

Let us determine the function

$$F^{\mathrm{ret}}(x) = \langle 0 \left| \frac{\delta j_\nu(0)}{\delta \bar{\psi}(x)} \right| \boldsymbol{p}s \rangle .$$

Its Fourier transform is

$$\tilde{F}^{\mathrm{ret}}(p'', p) = \int \exp [ip''x] F^{\mathrm{ret}}(x) \, \mathrm{d}x .$$

Let us also introduce the function $F^{\mathrm{adv}}(x)$ in the following way

$$F^{\mathrm{adv}}(x) = \langle 0 \left| \frac{\delta \Lambda(x)}{\delta A_\nu(0)} \right| \boldsymbol{p}s \rangle .$$

Note that the causality conditions require

$$F^{\mathrm{ret}}(x) = 0 , \qquad \text{if } x \lesssim 0,$$

and

$$F^{\mathrm{adv}}(x) = 0 , \qquad \text{if } x \gtrsim 0.$$

Let us consider the expression

$$\text{(A.3)} \qquad F(x) = F^{\mathrm{ret}}(x) - F^{\mathrm{adv}}(x) = \langle 0 \left| \frac{\delta j_\nu(0)}{\delta \bar{\psi}(x)} - \frac{\delta \Lambda(x)}{\delta A_\nu(0)} \right| \boldsymbol{p}s \rangle =$$

$$= i \langle 0 | [j_\nu(0), \Lambda(x)]_- | \boldsymbol{p}s \rangle ,$$

which can be written in the following way

$$\text{(A.4)} \qquad F(x) = i \sum_n \int \mathrm{d}\boldsymbol{k}_n \langle 0 | j_\nu(0) | n, \boldsymbol{k}_n \rangle \langle n, \boldsymbol{k}_n | \Lambda(x) | \boldsymbol{p}s \rangle +$$

$$- i \sum_n \int \mathrm{d}\boldsymbol{k}_n \langle 0 | \Lambda(x) | n, \boldsymbol{k}_n \rangle \langle n, \boldsymbol{k}_n | j_\nu(0) | \boldsymbol{p}s \rangle .$$

The Fourier transform of the first term in (A.4) is

$$\tilde{F}_1(p'', p) = i(2\pi)^4 \sum_n \int \mathrm{d}\boldsymbol{k}_n \, \delta(p'' + k_n - p) \langle 0 | j_\nu(0) | n, \boldsymbol{k}_n \rangle \langle n, \boldsymbol{k}_n | \Lambda(0) | \boldsymbol{p}s \rangle ,$$

where $k_n^0 > 0$, $k_n^2 \geqslant (2\mu)^2$. Therefore, $\tilde{F}_1$ is equal to zero if

$$\text{(A.5)} \qquad p_0 - p_0'' < 0 \qquad \text{or} \qquad (p - p'')^2 < (2\mu)^2 .$$

In the same way we can show that the second term in (A.4) is equal to zero if

$$p_0'' < 0 \qquad \text{or} \qquad p''^2 < (M+\mu)^2 . \tag{A.6}$$

Therefore

$$\tilde{F}^{\text{ret}}(p'', p) - \tilde{F}^{\text{adv}}(p'', p) = 0 , \tag{A.7}$$

if the inequalities

$$\begin{cases} p_0 - p_0'' < 0 & \text{or} \quad (p-p'')^2 < (2\mu)^2 , \\ p_0'' < 0 & \text{or} \quad p''^2 < (M+\mu)^2 , \end{cases} \tag{A.8}$$

are fulfilled simultaneously.

The inequalities (A.7) will be surely fulfilled if the inequalities

$$\begin{cases} p_0'' > p_0 & \text{or} \quad -\sqrt{(2\mu)^2 + \boldsymbol{p}^2} + p_0 < p_0'' < \sqrt{(2\mu)^2 + \boldsymbol{p}^2} + p_0 , \\ p_0'' < 0 & \text{or} \quad -(M+\mu) < p_0'' < (M+\mu) , \end{cases} \tag{A.9}$$

are fulfilled simultaneously.

But this means that

$$p_0 - \sqrt{(2\mu)^2 + \boldsymbol{p}^2} < p_0'' < M + \mu . \tag{A.10}$$

Thus, if (A.10) is valid, then (A.7) is also verified. Now we can make use of the following theorem (7).

Theorem. – Let the generalized functions $F^{\text{ret}}(x)$ and $F^{\text{adv}}(x)$ possess the properties

$$\begin{aligned} F^{\text{ret}}(x) &= 0 , && \text{if} \quad x \lesssim 0 , \\ F^{\text{adv}}(x) &= 0 , && \text{if} \quad x \gtrsim 0 , \\ \tilde{F}^{\text{ret}}(p) - \tilde{F}^{\text{adv}}(p) &= 0 . && \text{if} \quad \alpha < p_0 < \beta . \end{aligned}$$

Then there exists a function $\tilde{\Phi}(q)$ depending on the 4-vector q, analytical in the region

$$|\operatorname{Im} \boldsymbol{q}| < |\operatorname{Im} \sqrt{(q^0 - \alpha)(q^0 - \beta)}| , \tag{A.11}$$

and such that for all real $q = p$ where $\alpha < p_0 < \beta$ the following equalities are verified

$$\tilde{\Phi}(p) = \tilde{F}^{\text{ret}}(p) = \tilde{F}^{\text{adv}}(p) . \tag{A.12}$$

In our case

$$\boldsymbol{p}'' = \lambda \boldsymbol{e} = \sqrt{\frac{\tau^2 + 4M^2(-\tau - \boldsymbol{p}^2)}{4(M^2 + \boldsymbol{p}^2)}}\, \boldsymbol{e} ;$$

$$p_0'' = \frac{2M^2 - \tau}{2\sqrt{M^2 + \boldsymbol{p}^2}} ;$$

$$\alpha = \sqrt{M^2 + \boldsymbol{p}^2} - \sqrt{(2\mu)^2 + \boldsymbol{p}^2} ; \qquad \beta = M + \mu .$$

Thus, the regularity region is determined by the inequalities:

$$\text{(A.13)} \qquad \left| \operatorname{Im} \sqrt{\frac{\tau^2 + 4M^2(-\tau - \boldsymbol{p}^2)}{4(M^2 + \boldsymbol{p}^2)}} \right| < \\ < \operatorname{Im} \sqrt{\sqrt{(2\mu)^2 + \boldsymbol{p}^2} - \left(\frac{\tau + 2\boldsymbol{p}^2}{2\sqrt{M^2 + \boldsymbol{p}^2}}\right)\left(\frac{2M^2 - \tau}{2\sqrt{M^2 + \boldsymbol{p}^2}} - M - \mu\right)} .$$

Hitherto it was supposed that $\tau < -\boldsymbol{p}^2$.

It is easily seen that (A.13) is fulfilled for $\tau = 0$ which we are interested in, only if

$$\boldsymbol{p}^2 < \sigma_1 \mu^2, \qquad \sigma_1 = 2{,}8 .$$

From relativistic invariance and the conservation of current

$$k_\nu \langle p''s'' | j_\nu(0) | ps \rangle = 0 , \qquad k = p'' - p ,$$

it follows that (A.1) can be represented in the form [8]:

$$\text{(A.14)} \qquad \bar{u}(p'') \left\{ \frac{1 + \tau_3}{2} e\gamma_\nu F(k^2) + \frac{1}{2} \hat{\mu} [(k \cdot \gamma), \gamma_\nu] \right\} u(p),$$

where

$$\hat{\mu} = \mu_p \frac{1 + \tau_3}{2} G_p(k^2) + \mu_n \frac{1 - \tau_3}{2} G_n(k^2) , \qquad k^2 = \tau,$$

and F, G_p, G_n are relativistic invariant form-factors. Since $F^{\text{ret}}(p'', p)$ is an analytical function of τ in the region $-V < \tau < \varrho\mu^2$ then the expression

$$\left\{ \frac{1 + \tau_3}{2} e\gamma_\nu F(\tau) + \frac{\hat{\mu}}{2} [(k\gamma), \gamma_\nu] \right\} u(p),$$

is also an analytical function of τ in this region. Thus, the coefficients $F(\tau)$, $G_p(\tau)$ and $G_n(\tau)$ are analytical in τ. We can easily show, that the form-factors $F(\tau)$, $G_p(\tau)$ and $G_n(\tau)$ are real functions of τ if we remember that the current operator is hermitian and note that at $\tau < -\boldsymbol{p}^2$ the momentum p'' entering the spectrum $\bar{u}''(p'')$ describes the real particle.

In the following we make use of the theorem:

If the function $\Phi(\tau)$ is an analytical function of τ in the region G and for some interval of the real axis which is in G it is real, then it is real for the whole part of the real axis belonging to the region G.

---

[8] I. L. Foldy: *Phys. Rev.*, **87**, 688 (1952).

Since the point $\tau = 0$ is in the region of analyticity (A.13), then $F(\tau)$, $G_p(\tau)$ and $G_n(\tau)$ are real up to $\tau = 0$. At $\tau = 0$ $F(0) = G_p(0) = G_n(0) = 1$ [9].

Hence the physical meaning of the coefficients e, μ_p and μ_n is quite obvious: the first corresponds to a renormalized electrical charge and the second and third ones correspond to the anomalous magnetic moments of proton and neutron respectively.

[9] A. A. LOGUNOV, A. N. TAVKHELIDZE and L. D. SOLOVIEV: *Nucl. Phys.*, **4**, 427 (1957).

RIASSUNTO (*)

Si dà una dimostrazione esatta delle relazioni di dispersione per lo scattering Compton su nucleoni. La dimostrazione si ottiene seguendo un metodo suggerito da BOGOLJUBOV; si prendono in considerazione solo interazioni forti.

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Generalized Dispersion Relations.

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Summary. — General dispersion relations for reactions $a+b \to a'+c+d$ have been obtained. The possible cases of $(\gamma+p \to 2\gamma+p)$ scattering have been indicated, for which dispersion relations do not contain unobservable range of energy.

Introduction.

Dispersion relations for reactions with varying number of particles (with one nucleon and one boson in the initial state of the reaction and with one nucleon and two bosons in the final state) have been obtained in the works [1-4] for those cases of scattering in which boson energies in the final state of the reaction are equal.

In the work [2,3] strict proof of dispersion relations has been given for the above mentioned cases of scattering under absence of an unobservable region. The method of getting dispersion relations for the an reactions $a+b \to a'+c+d$ is indicated in this paper without additional assumptions about the equality of energy of the final bosons.

When writing dispersion relations it turned out more suitable as was mentioned in the work [5], to fix relations of final boson energies. Analytic properties of the amplitude for the general scattering case (on the example of double Compton effect) are being considered in the work [6].

[1] A. A. Logunov: *Dokl. Akad. Nauk SSSR* (in print).
[2] A. L. Logunov and A. N. Tavkhelidze: *Dokl. Akad. Nauk SSSR* (in print).
[3] A. A. Logunov and A. N. Tavkhelidze: *Nuclear Physics* (in print).
[4] A. A. Logunov, A. N. Tavkhelidze and N. A. Černikov: *Zeits. f. Naturf.* (in print).
[5] J. C. Polkinghorne: *Nuovo Cimento*, **4**, 216 (1956).
[6] A. A. Logunov, S. M. Bilenky and A. N. Tavkhelidze: *Nauchnye Dokl. Vysšei Školy* (in print).

1. – Kinematics of the process.

Let us define the momentum of the initial and final nucleons: p and p', and the momentum of the initial and final bosons: q and q', q'' correspondingly.

Accordingly to the laws of conservation of energy and momentum of the system we get:

$$p + q = p' + q' + q'' . \tag{1.1}$$

Further on we shall work in the system of:

$$\boldsymbol{p} + \boldsymbol{p}' = 0 . \tag{1.2}$$

When getting dispersion relations for reactions of the considered form it is very suitable (as we shall see further) to fix by arbitrary means the ratio of the produced boson energies.

$$\frac{q_0'}{q''} = \frac{\nu'}{\nu''} , \tag{1.3}$$

where ν' and ν'' are arbitrary fixed numbers.

Without limiting the generality it is suitable to take:

$$\nu'\nu'' = 1 . \tag{1.4}$$

Instead of the vectors q' and q'' we introduce vectors defined as:

$$\begin{cases} Q &= \frac{1}{2}\{\nu''(1-\eta)q' + \nu'(1+\eta)q''\} , \\ \Delta &= \frac{1}{2}(\nu''q' - \nu'q'') , \\ Q\Delta = 0, \end{cases} \tag{1.5}$$

where

$$\eta = +\frac{1}{4}\left[\nu''^2\left(\frac{m_1}{\Delta}\right)^2 - \nu'^2\left(\frac{m_2}{\Delta}\right)^2\right] ,$$

m_1, m_2, masses of produced bosons.

In our choice of vectors Q and Δ one can easily see that they are orthogonal and vector Δ is space-like, and vector Q time-like. Fixing the square of the 4-vector Q

$$Q^2 = m_Q^2 , \tag{1.6}$$

laws of conservation of energy and momentum (1.1) in our system can be written in the following form:

$$\begin{cases} \boldsymbol{q} - 2\alpha\boldsymbol{Q} = -2\boldsymbol{p} + 2\beta\boldsymbol{\Delta}\,, \\ (\boldsymbol{q} - 2\alpha\boldsymbol{Q})(\boldsymbol{q} + 2\alpha\boldsymbol{Q}) = \gamma^2 M_Q^2\,, \end{cases} \tag{1.7}$$

where

$$2\alpha = \frac{\nu' + \nu''}{\nu'\nu''}\,, \qquad \beta = -\frac{1}{2}\left[\frac{(1-\eta)}{\nu'} - \frac{(1+\eta)}{\nu''}\right].$$

From this after an elementary calculation we get:

$$\begin{cases} \boldsymbol{q} = \lambda\boldsymbol{e} - (1+\varepsilon_1)\boldsymbol{p} - \varepsilon_2\boldsymbol{n} + \beta\boldsymbol{\Delta}\,, \\ 2\alpha\boldsymbol{Q} = \lambda\boldsymbol{e} + (1-\varepsilon_1)\boldsymbol{p} - \varepsilon_2\boldsymbol{n} - \beta\boldsymbol{\Delta}\,, \end{cases} \tag{1.8}$$

where $\boldsymbol{e}$, $\boldsymbol{n}$ and $\boldsymbol{p}/|\boldsymbol{p}|$ are two orthogonal unit vectors (vector $\boldsymbol{e}$ lies in the plane $(\boldsymbol{p}, \boldsymbol{\Delta})$), and

$$\lambda^2 = 4\alpha^2 E'^2 - 4\alpha^2 m_Q^2 - (1-\varepsilon_1)^2\boldsymbol{p}^2 - \varepsilon_2^2 + 2\beta(1-\varepsilon_1)(\boldsymbol{p}\boldsymbol{\Delta}) - 2\alpha\beta\varepsilon_2(\boldsymbol{\Delta}\boldsymbol{n})\,. \tag{1.9}$$

By E' we define the zero component of vector Q. The values ε_1 and ε_2 in expression (1.8) are determined from relations (1.5) and (1.7). Further on we shall consider double Compton-effect $(\gamma + p \to 2\gamma + p)$. In this case formulas (1.5)–(1.9). considerably simplify $(\eta = 0)$ and have the following form:

$$\begin{cases} Q^2 = -\Delta^2 = m_Q^2\,, \\ q' = \dfrac{Q+\Delta}{\nu''}\,, \qquad q'' = \dfrac{Q-\Delta}{\nu'}\,, \\ q' = \lambda\boldsymbol{e} - (1+\varepsilon_1)\boldsymbol{p} - \varepsilon_2\boldsymbol{n} + \beta\boldsymbol{\Delta}\,, \\ 2\alpha\boldsymbol{Q} = \lambda\boldsymbol{e} + (1-\varepsilon_1)\boldsymbol{p} - \varepsilon_2\boldsymbol{n} - \beta\boldsymbol{\Delta}\,, \end{cases} \tag{1.10}$$

where

$$\begin{cases} \varepsilon_1\boldsymbol{p}^2 = [m_Q^2 + \beta(\boldsymbol{p}\boldsymbol{\Delta})]\,, \\ \varepsilon_2(\boldsymbol{\Delta}\boldsymbol{n}) = (1-\varepsilon_1)(\boldsymbol{p}\boldsymbol{\Delta}) - \beta m_Q^2\,. \end{cases} \tag{1.11}$$

As a variable in which we shall consider analytic properties of reactions amplitude, we take the variable ω, which is defined by the following way:

$$q_0' = \nu'\omega\,, \qquad q_0'' = \nu''\omega\,, \qquad E' = \omega\,. \tag{1.12}$$

It is clear, that variables $\boldsymbol{p}$, $\boldsymbol{\Delta}$, ω are independent. We should point out, that in formulas (1.10)–(1.11) in the vector expansion $\boldsymbol{q}'$, $\boldsymbol{q}''$, $\boldsymbol{q}$ in direction of $\boldsymbol{e}$, $\boldsymbol{p}$, $\boldsymbol{n}$ the components on $\boldsymbol{p}$ and $\boldsymbol{n}$ are fixed and do not depend on ω and the component in the direction of vector $\boldsymbol{e}$ is a function of ω.

The boundary between observable (real momentum: the energy is positive) and unobservable regions is determined by the variable λ: thus if the variable a is such, that

$$\lambda^2 > 0\,, \tag{1.13}$$

in this case momenta are real, and we have an observable region; if

$$\gamma^2 < 0 \tag{1.14}$$

we can see easily, that vectors $\boldsymbol{q}'$, $\boldsymbol{q}''$ and $\boldsymbol{q}$ become complex, and we have an unobservable region.

Since λ is a function of ω, we can determine the threshold energy ω_p from the condition:

$$\lambda^2 = 0\,. \tag{1.15}$$

Substituting (1.11) in (1.9) and after elementary transformations we get:

$$\lambda^2 = (\nu' + \nu'')^2\omega_p - (\nu' + \nu'')m_Q^2 - \frac{1}{\sin^2(\boldsymbol{p}\boldsymbol{\Delta})}\left[|\boldsymbol{p}|\left(1 - \frac{m_Q^2}{\boldsymbol{p}^2}\right) - (\nu' - \nu'')\frac{\boldsymbol{\Delta}\boldsymbol{p}}{|\boldsymbol{p}|}\right]^2, \tag{1.16}$$

from which the threshold energy equals:

$$(\nu' + \nu'')^2\omega_p^2 = (\nu' + \nu'')m_Q^2 + \frac{1}{\sin^2(\boldsymbol{p}\boldsymbol{\Delta})}\left[|\boldsymbol{p}|\left(1 - \frac{m_Q^2}{\boldsymbol{p}^2}\right) - (\nu' - \nu'')\cdot\frac{(\boldsymbol{\Delta}\boldsymbol{p})}{|\boldsymbol{p}|}\right]^2. \tag{1.17}$$

2. – Analysis of the antihermitian part of the amplitude process.

As it was shown in the work [3], the antihermitian part of amplitude can be written down in the following form:

$$A_{q,q',q''} = -\frac{i}{\sqrt{8q_0q_0'q_0''}}\int \exp\left[i(q'x' + q''x'' - qx)\right]\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}x''\cdot$$
$$\cdot\langle p', s'\left|\frac{\delta^2 j_Q(x)}{\delta\varphi_{Q'}(x')\,\delta\varphi_{Q''}(x'')} + \frac{\delta^2 A_Q(x)}{\delta\varphi_{Q'}(x')\delta\varphi_{Q''}(x'')}\right| p, s\rangle\,, \tag{2.1}$$

where

$$j_Q(x) = i\frac{\delta S}{\delta\varphi_Q(x)}S^+, \qquad aA_Q(x) = -S_{Q}^{+}(x)S\,.$$

Using the property of relativistic invariance of matrix elements, and making the assumption, that the operator of energy-momentum has a complete system of state vectors we have:

$$(2.2)\quad A_{q,q',q''} = \frac{2(2\pi)^4}{\sqrt{8q_0q_0'q''}}\,\delta(p+q-p'-q'-q'')A_{\alpha,\omega}(q,q',q''),$$

$$A_{\alpha,\omega}(q,q',q'') = \pi\sum\{V_{q'}(n,p')D_{q,q''}(p,n)\,\delta(p_0'+q_0'-\sqrt{M_n^2+(\boldsymbol{p}'+\boldsymbol{q}')^2}) -$$

$$-\delta(p_0-q_0'-\sqrt{M_n^2+(\boldsymbol{p}-\boldsymbol{q}')^2})D_{q,q''}(n,p')V_{q''}(p,n) +$$

$$+\delta(p_0+q_0''-\sqrt{M_n^2+(\boldsymbol{p}'+\boldsymbol{q}'')^2})V_{q''}(n,p')D_{q,q'}(p,n) -$$

$$-\delta(p_0-q_0''-\sqrt{M_n^2+(\boldsymbol{p}-\boldsymbol{q}'')^2})D_{q,q'}(n,p')V_{q''}(p,n) +$$

$$+\delta(p_0-q_0-\sqrt{M_n^2-(\boldsymbol{p}+\boldsymbol{q})^2})D_{-q,q''}(n,p')V_q(p,n) -$$

$$-\delta(p'-q_0-\sqrt{M_0^2+(\boldsymbol{p}'-\boldsymbol{q})^2})V_q(n,p')D_{-q',q''}(p,n).$$

Let us consider the δ function of the expression (2.2). It is easy to show, that the singularities of these functions δ (in the system where the sum of momentum of nucleons before and after reaction is equal zero) will be at the energies:

$$(2.3)\quad \begin{cases} \pm 2\nu' p_0\omega = M_n^2 - M^2 + \dfrac{2}{\nu''}\cdot\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu'}{\nu'+\nu''}\boldsymbol{P}^2 - \dfrac{4}{\nu'+\nu''}(\boldsymbol{p}\cdot\boldsymbol{\Delta}), \\ \pm 2\nu'' p_0\omega = M_n^2 - M^2 + \dfrac{2}{\nu'}\cdot\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu''}{\nu'+\nu''}\boldsymbol{P}^2 + \dfrac{4}{\nu'+\nu''}(\boldsymbol{p}\cdot\boldsymbol{\Delta}), \\ \pm 2(\nu'+\nu'')p_0\omega = M_n^2 - M^2 - 2\boldsymbol{P}^2 - 2m_Q^2. \end{cases}$$

On the basis (2.3) the δ-singularities of the functions with $M_n = M$ (one nucleon intermediate state) will be at the energy:

$$(2.4)\quad \begin{cases} \pm 2\nu' p_0\omega_1 = \dfrac{2}{\nu''}\cdot\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu'}{\nu'+\nu''}\boldsymbol{P}^2 - \dfrac{4}{\nu'+\nu''}(\boldsymbol{p}\boldsymbol{\Delta}), \\ \pm 2\nu'' p_0\omega_2 = \dfrac{2}{\nu'}\cdot\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu''}{\nu'+\nu''}\boldsymbol{P}^2 + \dfrac{4}{\nu'+\nu''}(\boldsymbol{p}\boldsymbol{\Delta}), \\ \pm 2(\nu'+\nu'')p_0\omega_3 = -2\boldsymbol{P}^2 - 2m_Q^2. \end{cases}$$

Assuming that between M and $M+\mu$ (μ, mesons mass) no bound state of meson-nucleon system exists, putting in (2.3) $M_n \geqslant M+\mu$ we obtain, that

the continuous spectrum lays in the region, where:

$$(2.5)\qquad \begin{cases} \pm 2\nu' p_0 \omega_{1c} = 2M_\mu + \mu^2 + \dfrac{2}{\nu''}\cdot\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu'}{\nu'+\nu''}\,\boldsymbol{p}^2 - \dfrac{4}{\nu'+\nu''}\,(\boldsymbol{p}\boldsymbol{\Delta})\,, \\ \pm 2\nu' p_0 \omega_{2c} = 2M_\mu + \mu^2 + \dfrac{2}{\nu'}\dfrac{m_Q^2}{\nu'+\nu''} - \dfrac{2\nu''}{\nu'+\nu''}\,\boldsymbol{p}^2 + \dfrac{4}{\nu'+\nu''}\,(\boldsymbol{p}\boldsymbol{\Delta})\,, \\ \pm 2(\nu'+\nu'')p_0\omega_c = -2\boldsymbol{p}^2 - 2m_Q^2 + 2M_\mu + \mu^2\,. \end{cases}$$

We assume, that (for example we take $\nu' > \nu''$):

$$(2.6)\qquad \begin{cases} m_Q^2 < \frac{1}{4}(M_\mu + \frac{1}{2}\mu^2)\,, \\ \boldsymbol{p}^2 < \frac{1}{4}(M_\mu + \frac{1}{2}\mu^2)\,, \\ pm_Q < \frac{1}{4}\nu''(M_\mu + \frac{1}{2}\mu^2)\,. \end{cases}$$

then the continuous spectrum is above the one nucleon poles and the energy spectrum of the antihermitian part of the amplitude can be schematically designed in the following way:

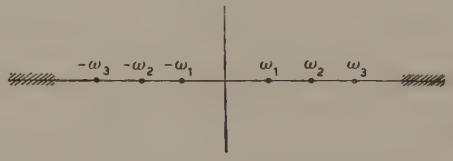

Fig. 1.

$$(2.7)\qquad \pm(\nu'+\nu'')P_0\omega_c = -\boldsymbol{p}^2 - m_Q^2 + M_\mu + \tfrac{1}{2}\mu^2\,.$$

Instead of inequalities (2.6) we take stronger inequalities of the form:

$$(2.8)\qquad \begin{cases} m_Q^2 < \dfrac{\nu''^2}{4}\left(M_\mu + \dfrac{1}{2}\mu^2\right), \\ \boldsymbol{p}^2 < \dfrac{1}{4}\left(M_\mu + \dfrac{1}{2}\mu^2\right), \end{cases}$$

or

$$(2.9)\qquad \begin{cases} \boldsymbol{p}^2 < \dfrac{1}{4}\dfrac{\nu''}{\nu'}\left(M_\mu + \dfrac{1}{2}\mu^2\right), \\ m_Q^2 < \dfrac{1}{4}\quad\left(M_\mu + \dfrac{1}{2}\mu^2\right). \end{cases}$$

(*) In case $\nu' > \nu''$, ν' substitutes ν'' in formulas (2.6).

As it was mentioned in Sect. 1, an observed energy region ω continues from ω_0 (threshold energy). The unobservable region in dispersion relations will be absent if the continuous spectrum starts above the threshold of the process

$$\omega_c \geqslant \omega_0 . \tag{2.10}$$

The condition (2.10) imposes some limitations to the possible values $\boldsymbol{p}$ and m_0^2. Substituting (1.17) and (2.10) we can get:

$$D^2 > \left[1 + \frac{1}{4}(\nu' - \nu'')\right] 4m_0^2 + \frac{1}{\sin^2 (\boldsymbol{p}\Delta)} \left[\frac{\boldsymbol{p}^2 - m_0^2}{|\boldsymbol{p}|} - (\nu' - \nu'') \cos (\boldsymbol{p}\Delta)\right]^2 , \tag{2.11}$$

where

$$D = \frac{M_\mu + \frac{1}{2}\mu^2 - \boldsymbol{p}^2 - m_0^2}{\sqrt{\boldsymbol{p} + M^2}} . \tag{2.12}$$

It is easy to find, that if

$$\frac{(M_\mu + \frac{1}{2}\mu^2 - \boldsymbol{p}^2 - m_0^2)^2}{(\boldsymbol{p}^2 + M^2)} \geqslant \frac{(\boldsymbol{p}^2 + m_0^2)^2}{\boldsymbol{p}^2} , \tag{2.13}$$

there are always such values $(\nu' - \nu'')$ which lay between values $(\nu' - \nu'')_1$ and $(\nu' - \nu'')_2$ which are equal:

$$(\nu' - \nu'')_{1,2} = \frac{\cos \alpha [(\boldsymbol{p}^2 - m_0^2)/|\boldsymbol{p}|] \pm \sin \alpha \sqrt{D^2 - (\boldsymbol{p}^2 + m_0^2)/\boldsymbol{p}^2}}{2m_0} , \tag{2.14}$$

for which (2.10) is satisfied.

From the analysis of the condition (2.13) it results that, if

$$m_0^2 \leqslant \tfrac{1}{4}\mu^2 \tag{2.15}$$

possible values of the momentum $\boldsymbol{p}^2$ always exist for which the inequality (2.13) is satisfied. We should note that if $m_0^2 = \frac{1}{4}\mu^2$ there is only one momentum value

$$\boldsymbol{p}_c^2 = \frac{1}{4}\mu^2 \frac{M}{M + \mu} , \tag{2.16}$$

and in the case $m_0^2 < \frac{1}{4}\mu^2$ there exist some interval $\boldsymbol{p}^2$ between values, which includes point $\boldsymbol{p}_c$ for which (2.13) is satisfied. Comparing (2.12) with (2.13) we can easily see that the condition (2.13) is a condition for the absence in dispersion relations of the unobservable region for those cases of scattering, in which the energies of produced photons are equal $(\nu' = \nu'' = 1)$ and the angle $\widehat{\hat{p}\hat{\Delta}}$ equals 4/2.

Thus, if in dispersion relations the unobservable region is absent for those scattering cases, in which photon energies are equal ($\nu' = \nu'' = 1$), and angle $\widehat{p\Delta}$ equals the angle in the general case, for any angle (ν') always exists some interval of the values for which the unobservable region in dispersion relations is also absent.

In the case of equal energies ($\nu' = \nu'' = 1$) the unobservable region is absent if the following condition is satisfied:

$$\operatorname{cotg} q^2 \leqslant \frac{\boldsymbol{p}^2[D^2 - ((\boldsymbol{p} + m_Q^2)/|\boldsymbol{p}|)^2]}{(\boldsymbol{p}^2 - m_Q^2)^2}. \tag{2.17}$$

This condition can be easily obtained both: directly from (2.12) and from the demand that point $\nu' - \nu' = 0$ (equal energies) lay in intervals between the radicals $(\nu' - \nu'')_1$ and $(\nu' - \nu'')_2$.

3. – Dispersion relations.

In the works [2,3] a strict proof (in absence of unobservable region) of dispersion relations for those cases of scattering ($\gamma + p \to 2\gamma + p$) was given, in which photons in the final state of reaction have equal energies. One may show [6] that without any considerable change this proof may be extended to other cases of scattering (when photon energies are not equal) for which dispersion relations do not contain the unobservable region (condition (2.13) is satisfied). To write dispersion relations it is necessary to make a detailed analysis of the antihermitian part of the amplitude for the energy region $|E| < E_\pi$.

Since considering the region in the antihermitian part of the amplitude contributes only one nucleon state we have:

$$\begin{aligned} A_{\nu,\omega}(q, q', q'') = \pi \sum_{s''} \Big\{ & \frac{1}{\nu'}\left(1 + \frac{\nu'\omega_1}{p_0}\right) V_{q'}(p'', p') D_{q,q''} \delta(\omega - \omega_1) - \\ & - \frac{1}{\nu'}\left(1 + \frac{\nu'\omega_1}{p_0}\right) D_{q,q''}(p'', p') V_{q'}(p, p'') \delta(\omega - \omega_1) + \\ & + \frac{1}{\nu''}\left(1 + \frac{\nu''\omega_2}{p_0}\right) V_{q''}(p'', p') D_{q,q'}(p, p'') \delta(\omega - \omega_2) - \\ & - \frac{1}{\nu''}\left(1 + \frac{\nu''\omega_2}{p_0}\right) D_{q,q'}(p'', p') V_{q''}(p, p'') \delta(\omega + \omega_2) + \\ & + \frac{1}{\nu}\left(1 - \frac{\nu\omega_3}{p_0}\right) D_{-q',q''}(p'', p') V_q(p, p'') \delta(\omega + \omega_3) - \\ & - \frac{1}{\nu}\left(1 - \frac{\nu\omega_3}{p_0}\right) V_q(p'', p') D_{-q',q''}(p, p'') \delta(\omega - \omega_3) \Big\}, \end{aligned} \tag{3.1}$$

We should note that operators Vq' ,$Vq,''$ Vq are usual vertex operators (with one photon and two nucleon ends $q'^1 = 0$, $p^2 = M^2$, $p'^2 = M^2$) and operators $D_{q,q''}$, $D_{q,q'}$ etc., are 4-vertex (with two photons and two nucleon ends). Detailed consideration of operators V and D was performed in the work [3].

With the help of (3.1) we can easily calculate the contribution to the dispersion integral from the region $|E| < E_c$,

$$(3.2)\quad \frac{1}{\pi}\int_{-E_c}^{E_c} dE' \left(\frac{E-E_0}{E'-E_0}\right)^{n+1} \frac{S_{\pm} A_{\alpha,\omega}(E')}{(E'-E)} = \sum_{i=\alpha,\beta,\gamma} \left[\left(\frac{E_0-E}{E_0-E_i}\right)^{n+1} \frac{S_{\pm} R_i(\boldsymbol{p},\boldsymbol{\Delta},\boldsymbol{e})}{(E_i-E)} + \left(\frac{E_0-E}{E_0+E_i}\right) \frac{S_{\pm}\Omega_i(\boldsymbol{p},\boldsymbol{\Delta},\boldsymbol{e})}{(E_i+E)}\right],$$

where

$$(3.3)\quad \begin{cases} R_\alpha = -\dfrac{1}{\nu}\left(1-\dfrac{\nu\omega_3}{p_0}\right) V_q(p'', p') D_{-q',q''}(p, p''), \\ R_\beta = \dfrac{1}{\nu''}\left(1+\dfrac{\nu''\omega_2}{p_0}\right) V_q(p'', p') D_{q,q'}(p, p''), \\ R_\gamma = \dfrac{1}{\nu'}\left(1+\dfrac{\nu'\omega_1}{p_0}\right) V_{q'}(p'', p') D_{q,q''}(p, p''), \\ \Omega_\alpha = -\dfrac{1}{\nu}\left(1-\dfrac{\nu\omega_3}{p_0}\right) D_{-q',q''}(p'', p') V_q(p, p''), \\ \Omega_\beta = \dfrac{1}{\nu''}\left(1+\dfrac{\nu''\omega_2}{p_0}\right) D_{q,q'qq}(p'', p') V_{q''}(p, p''), \\ \Omega_\gamma = \dfrac{1}{\nu'}\left(1+\dfrac{\nu\omega_1}{p_0}\right) D_{q,q''}(p'', p') V_{q'}(p, p''), \end{cases}$$

$S_{\pm}$ operator of symmetrization or antisymmetrization on using (3.2) and analytic properties of amplitude of double Compton-effect (2.6) we may get the following dispersion relation:

$$(3.4)\quad S \pm D_{\alpha,\omega}(E, \nu', \boldsymbol{p}, \Delta) = \frac{(E-E_0)^{n+1}}{\pi} p\int_{|E'|\geq E_c} dE' \frac{S_{\pm} A_{\alpha,\omega}(E', \nu', \boldsymbol{p}, \boldsymbol{\Delta}, \boldsymbol{e})}{(E'-E)(E'-E_0)^{n+1}} + \pm \sum_{i=\alpha,\beta,\gamma} \left[\left(\frac{E_0-E}{E_0-E_i}\right)^{n+1} \frac{S_{\pm} R_i(\boldsymbol{p},\boldsymbol{\Delta},\boldsymbol{e})}{(E_i-E)} + \left(\frac{E_1-E}{E_0+E_i}\right)^{n+1} \frac{S_{\pm}\Omega(\boldsymbol{p},\boldsymbol{\Delta},\boldsymbol{e})}{(E_i+E)}\right] + P_n(E).$$

We should note, that E has the same sense as ω,

$$q'_0 = E\nu', \qquad q''_0 = E9''.$$

The negative region of energy in dispersion relations (3.4) may be excluded if we use the property of symmetrization of the antihermitian part of the amplitude:

$$A_{\alpha,\omega}(E, \nu', \boldsymbol{p}, \boldsymbol{\Delta}) = - P_{ss'} A^{*}_{\alpha,\omega}(-E, \nu', \boldsymbol{p}, \boldsymbol{\Delta}) ,$$

$P_{ss'}$ = operator of nucleon spin permutation.

* * *

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RIASSUNTO (*)

Si sono ottenute relazioni di dispersione generali per le reazioni $a+b \to a'+c+d$. Si sono indicati i possibili casi di scattering $\gamma+\mathrm{p} \to 2\gamma+\mathrm{p}$ pei quali le relazioni di dispersione non contengono zone d'energia inosservabili.

(*) *Traduzione a cura della Redazione.*

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On the Theory of Dispersion Relations.

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Summary. — The dispersion relations for the double Compton-effect in the case of absence of unobservable energy region are proved.

Introduction.

In the papers ([1-3]) the dispersion relations for reactions of the $(a+b \to a'+c+d)$ type were obtained. The analytical properties of the amplitude of the double Compton effect for final photons of equal energies were considered in the papers ([2,4]).

In this article the proof of the dispersion relations for the process $(\gamma+\mathrm{p} \to 2\gamma+\mathrm{p})$ in the case of absence of unobservable region is given. In this case the proof of the dispersion relations becomes simple and it consists of the following main stages. First of all, the retarded and advanced amplitudes of the process are constructed with the help of the causality principle in Bogoljubov's formulation ([5]); the first of them is connected with the direct process, and the second one with the inverse process.

These functions are determined for real energy values above the threshold of the process.

([1]) A. A. Logunov: *Dokl. Akad. Nauk SSSR* (in print).
([2]) A. L. Logunov and A. N. Tavkhelidze: *Nuclear Physics* (in print).
([3]) A. A. Logunov and A. N. Tavkhelidze: *Nuovo Cimento* (in print).
([4]) A. A. Logunov and A. N. Tavkhelidze: *Dokl. Akad. Nauk SSSR* (in print).
([5]) N. N. Bogoljubov: *Izv. Acad. Nauk SSSR, Serie Fiz.*, **19**, 237 (1955).
([6]) N. N. Bogoljubov and D. V. Shirkov: *Introduction to the Theory of Quantum Fields* (Moscow, 1957).

Then the functions (6) $\Phi^r(\varrho, E)$ and $\Phi^a(\varrho, E)$ are constructed. These functions are analytical in the upper and lower half-planes E respectively and coincide on some region of the real axis.

These functions determine a function, which is analytical in the whole plane of the complex variable E, except for the cut on the real axis.

On the edges of the cut the limit values of the functions $\Phi^r(\varrho, E)$ and $\Phi^a(\varrho, E)$ when ϱ tends to zero coincide with the retarded and advanced amplitudes respectively. Dispersion relations arise as a result of Cauchy's theorem applied to this function.

1. – The investigation of the antihermitian part of the amplitude of the process.

To obtain dispersion relations we need to study the properties of the antihermitian part of the amplitude. As usual, the antihermitian part is determined as the difference between the retarded and advanced amplitudes (2).

$$A_{\alpha,\omega}(q; q', q'') = \frac{1}{2i}\left(T^{\mathrm{ret}}_{\alpha,\omega}(q; q', q'') - T^{\mathrm{adv}}_{\alpha,\omega}(q; q', q'')\right), \tag{1.1}$$

here

$$T^{\mathrm{ret}}_{\alpha,\omega}(q; q', q'') = -\int \exp\left[i(q'x' + q''x'')\right] \mathrm{d}x'\,\mathrm{d}x'' \langle p', s' | \frac{\delta^2 j_\nu(0)}{\delta A_{\nu'}(x')\,\delta A_{\nu''}(x'')} | p s \rangle \,, \tag{1.2}$$

$$T^{\mathrm{adv}}_{\alpha,\omega}(q; q', q'') = \int \exp\left[i(q'x' + q''x'')\right] \mathrm{d}x'\,\mathrm{d}x'' \langle p', s' | \frac{\delta^2 A_\nu(0)}{\delta A_{\nu'}(x')\,\delta A_{\nu''}(x'')} | p, s \rangle \,,$$

q, q' and q'' are photon momenta in the initial and final states, $|p, s\rangle$ is the vector of the nucleon state, the indexes α and ω refer to the initial and final states and include all the quantum numbers, except the momenta of the particles. According to the causality principle the integration is carried out on the region:

$$x'_0 \geqslant |\boldsymbol{x}'|, \qquad x''_0 \geqslant |\boldsymbol{x}''|, \qquad x'_0,\ x''_0 \geqslant 0, \tag{1.3}$$

for the retarded amplitude and in the region:

$$|x'_0| \geqslant |\boldsymbol{x}'|, \qquad |x''_0| \geqslant |\boldsymbol{x}''|, \qquad x'_0,\ x''_0 \leqslant 0, \tag{1.4}$$

for the advanced one.

Inserting (1.2) in (1.1) and using the translational invariance of the matrix elements, as well as the completeness of the system of state-vectors of the energy-

momentum operator, we get:

$$
\begin{aligned}
(1.5)\quad A_{\nu,\omega}(q;q',q'') = \pi \sum_n \Big\{ & V_{q'}(n,p')D_{q;q''}(p,n)\,\delta(p_0'+q_0'-\sqrt{M_n^2+(\boldsymbol{p}'+\boldsymbol{q}')^2}) - \\
& - D_{q;q'}(n,p')V_{q''}(p,n)\,\delta(p_0-q_0'-\sqrt{M_n^2+(\boldsymbol{p}-\boldsymbol{q}')^2}) + \\
& + V_{q''}(n,p')D_{q;q'}(p,n)\,\delta(p_0'+q_0''-\sqrt{M_n^2+(\boldsymbol{p}'+\boldsymbol{q}'')^2}) - \\
& - D_{q;q''}(n,p')V_{q'}(p,n)\,\delta(p_0-q_0''-\sqrt{M_n^2+(\boldsymbol{p}-\boldsymbol{q}'')^2}) + \\
& + D_{-q';q''}(n,p')V_q(p,n)\,\delta(p_0+q_0-\sqrt{M_n^2+(\boldsymbol{p}+\boldsymbol{q})^2}) - \\
& - V_q(n,p')D_{-q';q''}(p,n)\,\delta(p_0'-q_0-\sqrt{M_n^2+(\boldsymbol{p}'-\boldsymbol{q})^2})\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
&(2\pi)^4\,\delta(p-q'-k_n)D_{q;q''}(n,p') = \\
&\qquad = \frac{1}{2}\int \exp\,[iq''x'' - iqx]\langle p',s'\,|\frac{\delta j_\nu(x)}{\delta A_{\nu''}(x'')} + \frac{\delta j_{\nu''}(x'')}{\delta A_\nu(x)}\,|\,p,s\rangle\,\mathrm{d}x\,\mathrm{d}x'',
\end{aligned}
$$

$$
V_{q'}(n,p') = \langle p',s'\,|\,j_{\nu'}(0)\,|\,n\rangle\,.
$$

It is covenient to carry on the investigation in the system of coordinates, where the sum of the nucleon momenta is equal to zero ($\boldsymbol{p}+\boldsymbol{p}'=0$). In this system the energy of the photon in the initial state is equal to the sum of the energies of the final photons:

$$
q_0 = q_0' + q_0''\,. \tag{1.6}
$$

We shall consider the dispersion relations in the case when the relation of the photon energies in the final state is fixed arbitrarily:

$$
\frac{q_0'}{q_0''} = \frac{\nu'}{\nu''}, \tag{1.7}
$$

where ν' and ν'' are connected according to the condition

$$
\nu'\nu'' = 1\,. \tag{1.8}
$$

Let us introduce the vectors Q and Δ instead of q' and q'':

$$
\begin{cases}
Q = \frac{1}{2}(\nu''q' + \nu'q''), \\
\Delta = \frac{1}{2}(\nu''q' - \nu'q'').
\end{cases} \tag{1.9}
$$

Taking into account that $q'^2 = q''^2 = 0$ we have

$$Q\Delta = 0 . \tag{1.10}$$

It is easy to see that Q is a time-like vector, and Δ is a space-like vector.

Fixing the square of the Q vector in the following way

$$Q^2 = m_Q^2 . \tag{1.11}$$

we reduce the kinematics of the process under observation to the case of two particles in the final state, and the mass of one of them is variable. Using the laws of conservation of energy-momentum and taking into consideration (1.9) and (1.11) we obtain:

$$\left\{\begin{array}{l} \boldsymbol{q} = \lambda \boldsymbol{e} - (1+\varepsilon_1)\boldsymbol{p} - \varepsilon_2 \boldsymbol{n} + \beta \boldsymbol{\Delta} , \\ 2\alpha \boldsymbol{Q} = \lambda \boldsymbol{e} + (1-\varepsilon_1)\boldsymbol{p} - \varepsilon_2 \boldsymbol{n} - \beta \boldsymbol{\Delta} , \\ \text{and} \\ q' = \dfrac{Q+\Delta}{\nu''} , \qquad q'' = \dfrac{Q-\Delta}{\nu'} . \end{array}\right. \tag{1.12}$$

as well, where

$$\left\{\begin{array}{ll} \varepsilon_1 \boldsymbol{p}^2 = m_Q^2 + \beta(\boldsymbol{p}\boldsymbol{\Delta}) , & \varepsilon_2(\boldsymbol{\Delta}\boldsymbol{n}) = (1-\varepsilon_1)(\boldsymbol{p}\boldsymbol{\Delta}) - \beta m_Q^2 , \\ 2\alpha = (\nu' + \nu'') , & 2\beta = (\nu' - \nu'') , \end{array}\right. \tag{1.13}$$

$\boldsymbol{e}$, $\boldsymbol{n}$ and $\boldsymbol{p}/|\boldsymbol{p}|$ are mutually orthogonal unit vectors (the vector $\boldsymbol{n}$ is in the plane $(\boldsymbol{p}, \boldsymbol{\Delta})$). We shall consider the analytical properties of the amplitude with respect to the variable $Q_0 E$. As can be easily seen from (1.12) the quantity λ is connected with E in the following way:

$$\lambda = (\nu' + \nu'')\sqrt{E^2 - E_p^2} , \tag{1.14}$$

where E_p is the threshold of the reaction:

$$(\nu' + \nu'')^2 E_p^2 = (\nu' + \nu'')^2 m_Q^2 + (1 + \operatorname{ctg}^2(\widehat{\boldsymbol{p}\boldsymbol{\Delta}}))\left[\boldsymbol{P}^2 - \frac{m_Q^2}{|\boldsymbol{p}|} - (\nu' - \nu'')\frac{(\boldsymbol{\Delta}\boldsymbol{p})}{|\boldsymbol{p}|}\right]^2 . \tag{1.15}$$

Note that in our system $\boldsymbol{p}$, $\boldsymbol{\Delta}$, E, ν' are independent variables.

Let us now consider the δ-singularities of the antihermitian part of the amplitude (1.5). We will consider $\boldsymbol{Q}$ and E as independent variables; this

allows us to consider the expression (1.5) also in the region $|E| < E_p$. It is easy to see that for $M_n = M$ (one-nucleon state) δ functions in the expression (1.5) have singularities which are given by the solutions of the following equations:

$$(1.16)\quad \begin{cases} h_1(E, Q, \Delta, p) = [\nu'(Q^2 + \Delta^2) - 2p\Delta + 2\boldsymbol{pQ}]^2 - 4(\boldsymbol{p}^2 + M^2)E^2 = 0\,, \\ h_2(E, Q, \Delta, p) = [\nu''(Q^2 + \Delta^2) + 2p\Delta + 2\boldsymbol{pQ}]^2 - 4(\boldsymbol{p}^2 + M^2)E^2 = 0\,, \\ h_3(E, Q, \Delta, p) = [(\nu' + \nu'')^2 Q^2 + (\nu' - \nu'')^2\Delta^2 - 2(\nu' - \nu'')p\Delta + \\ \qquad + 2(\nu' + \nu'')(\boldsymbol{pQ})]^2 - 4(\nu' + \nu'')^2(\boldsymbol{p}^2 + M^2)E^2 = 0\,. \end{cases}$$

Supposing that in the interval $M \leqslant M_n \leqslant M + \mu$ there are no bound states and considering the lowest approximation in the electromagnetic interaction from the formula (1.5) we obtain the following values for the boundaries of continuous spectra:

$$(1.17)\quad \begin{cases} \pm\nu' E_{1c} = -p_0 + \sqrt{(M+\mu)^2 + (\nu'\boldsymbol{Q} - \boldsymbol{p})^2 + (\nu'\boldsymbol{\Delta} - \boldsymbol{p})^2 - \boldsymbol{p}^2}\,, \\ \pm\nu'' E_{2c} = -p_0 + \sqrt{(M+\mu)^2 + (\nu''\boldsymbol{Q} - \boldsymbol{p})^2 + (\nu''\boldsymbol{\Delta} - \boldsymbol{p})^2 - \boldsymbol{p}^2}\,, \\ \pm(\nu' + \nu'')E_{3c} = -p_0 + \sqrt{(M+\mu)^2 + [(\nu' + \nu'')\boldsymbol{Q} - \boldsymbol{p}]^2 + [(\nu' - \nu'')\boldsymbol{\Delta} - \boldsymbol{p}]^2 - \boldsymbol{p}^2}\,. \end{cases}$$

As is seen from these formulas the boundaries of continuous spectra depend on the vector $\boldsymbol{Q}$. Its minimum values are given by the following expressions:

$$(1.18)\quad \begin{cases} \pm\nu' E_{1c}^{\min} = -p_0 + \sqrt{(M+\mu)^2 + [\nu'\boldsymbol{\Delta} - \boldsymbol{p}]^2 - \boldsymbol{p}^2}\,, \\ \pm\nu'' E_{2c}^{\min} = -p_0 + \sqrt{(M+\mu)^2 + [\nu''\boldsymbol{\Delta} - \boldsymbol{p}]^2 - \boldsymbol{p}^2}\,, \\ \pm(\nu' + \nu'')E_{3c}^{\min} = -p_0 + \sqrt{(M+\mu)^2 + [(\nu' - \nu'')\boldsymbol{\Delta} - \boldsymbol{p}]^2 - \boldsymbol{p}^2}\,. \end{cases}$$

So, for any $\boldsymbol{Q}$ there are such $\boldsymbol{p}$, $\boldsymbol{\Delta}$ and ν', that:

$$(1.19)\qquad |E_{ic}^{\min}| \geqslant E_q > 0 \qquad (i = 1, 2, 3).$$

This means that on some region of the axis only one-nucleon poles can contribute to the antihermitian part.

Let us introduce the functions:

$$\tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(E, \boldsymbol{Q}, \boldsymbol{\Delta}, \boldsymbol{p})$$

determining them as follows:

$$(1.20)\qquad \tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(E, \boldsymbol{Q}, \boldsymbol{\Delta}, \boldsymbol{p}) = h_1 h_2 h_3 T^{\mathrm{ret}}_{\mathrm{adv}}(E, \boldsymbol{Q}, \boldsymbol{\Delta}, \boldsymbol{p})\,.$$

Then the function:

$$\tilde{A}_{\alpha,\omega}(E, \boldsymbol{Q}, \boldsymbol{\Delta}, p) = \frac{1}{2i}(\tilde{T}^{\mathrm{ret}}_{\alpha,\omega} - \tilde{T}^{\mathrm{adv}}_{\alpha,\omega}) \tag{1.21}$$

according to the relations (1.5) and (1.16) goes to zero in the interval $(-E_q, E_q)$. Introducing the variables Q and Λ, it is possible to write down the expression (1.21) as follows:

$$T^{\mathrm{ret}}_{\mathrm{adv}}(E, \boldsymbol{Q}, \boldsymbol{\Delta}) = \int \exp[iQx + i\Lambda y] T^{\mathrm{ret}}_{\mathrm{adv}}(x, y)\, \mathrm{d}x\, \mathrm{d}y\,, \tag{1.22}$$

where

$$\begin{cases} T^{\mathrm{ret}}(x, y) = -2^{-4}\langle p', s' | \dfrac{\delta^2 j_\nu(0)}{\delta A_{\nu'}((x+y)/2\nu')\, \delta A_{\nu''}((x-y)/2\nu'')} | p, s\rangle\,, \\ T^{\mathrm{adv}}(x, y) = +2^{-4}\langle p', s' | \dfrac{\delta^2 \Lambda_\nu(0)}{\delta A_{\nu'}((x+y)/2\nu')\, \delta A_{\nu''}((x-y)/2\nu'')} | p, s\rangle\,. \end{cases} \tag{1.23}$$

Multiplying the expression (1.22) by the product of the functions (1.16) we obtain:

$$\tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(E, \boldsymbol{Q}, \boldsymbol{\Delta}) = \int \exp[iQx + i\Lambda y] \tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(x, y)\, \mathrm{d}x\, \mathrm{d}y\,, \tag{1.24}$$

where

$$\tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(x, y) = \hat{h}_1\, \hat{h}_2\, \hat{h}_3\, T^{\mathrm{ret}}_{\mathrm{adv}}((x, y)$$

$$\begin{cases} \hat{h}_1 = [-\nu'(\Box_x^2 + \Box_y^2) + 2i(\boldsymbol{p}\nabla_y)^{\wedge} + 2i(\boldsymbol{p}\nabla_x)]^2 + 4(\boldsymbol{p}^2 + M^2)\, \dfrac{\partial^2}{\partial x_0^2}\,, \\ \hat{h}_2 = [-\nu''(\Box_x^2 + \Box_y^2) - 2i(\boldsymbol{p}\nabla_y) + 2i(\boldsymbol{p}\nabla_x)]^2 + 4(\boldsymbol{p}^2 + M^2)\, \dfrac{\partial^2}{\partial x_0^2}\,, \\ \hat{h}_3 = [-(\nu' + \nu'')^2 \Box_x^2 - (\nu' - \nu'')\Box_y^2 + 2i(\nu' - \nu'')(\boldsymbol{p}\nabla_y) + \\ \qquad + 2i(\nu' + \nu'')(\boldsymbol{p}\nabla_x)]^2 + 4(\nu' + \nu'')^2(\boldsymbol{p}^2 + M^2)\, \dfrac{\partial^2}{\partial x_0^2}\,. \end{cases} \tag{1.25}$$

Substituting (1.24) into (1.21) we get:

$$\tilde{A}_{\alpha,\omega}(E, \boldsymbol{Q}, \boldsymbol{\Delta}) = \int \exp[iQx + i\Lambda y] \tilde{A}_{\alpha,\omega}(x, y)\, \mathrm{d}x\, \mathrm{d}y\,. \tag{1.26}$$

As E and $\boldsymbol{Q}$ are supposed to be independent, using Fourier transformation it is possible to rewrite the expression (1.26) as follows:

$$(1.27)\quad \frac{1}{(2\pi)^3}\int \exp\left[i\boldsymbol{Q}\boldsymbol{x}\right]\tilde{A}_{\nu,\omega}(E,\boldsymbol{Q},\boldsymbol{\Delta})\,\mathrm{d}\boldsymbol{Q} = \int \exp\left[iEx_0 + i\boldsymbol{\Delta}y\right]\tilde{A}_{\nu,\omega}(x_0\boldsymbol{x}, y)\,\mathrm{d}x_0\,\mathrm{d}y\ .$$

Taking into account, that $\tilde{A}_{\nu,\omega}(E,\boldsymbol{Q},\boldsymbol{\Delta})$ goes to zero in the interval we obtain:

$$(1.28)\quad \begin{cases} \int \exp\left[iEx_0 + i\boldsymbol{\Delta}y\right]\tilde{A}_{\alpha,\omega}(x_0,\boldsymbol{x},y)\,\mathrm{d}x_0\,\mathrm{d}y = 0\ , \\ |E| < E_q\ , \qquad \operatorname{Im} E = 0\ . \end{cases}$$

Let us introduce the functions $\Phi^r(E,\boldsymbol{e},\boldsymbol{p},\boldsymbol{\Delta},\varrho)$ and $\Phi^a(E,\boldsymbol{e},\boldsymbol{p},\boldsymbol{\Delta},\varrho)$ in the following way:

$$(1.29)\quad \Phi^{r,a}(E,\boldsymbol{e},\boldsymbol{p},\boldsymbol{\Delta},\varrho) = S_{\pm}\int \tilde{T}^{\mathrm{ret}}_{\mathrm{adv}}(x,y)\exp\left[iEx_0 - i\boldsymbol{e}\boldsymbol{x}\sqrt{E^2-E_p^2} - \right.$$
$$\left. - i(\boldsymbol{p}\boldsymbol{x})\eta - i(\boldsymbol{\Delta}\boldsymbol{x})\xi - i\boldsymbol{\Delta}\boldsymbol{y} - \varrho\boldsymbol{x}^2\right]\mathrm{d}x\,\mathrm{d}y\ ;$$

where

$$\varrho > 0\ , \qquad \eta = \frac{1}{\nu' + \nu''}\left[1 - \varepsilon_1 + \frac{\boldsymbol{\Delta}\boldsymbol{p}}{\boldsymbol{\Delta}\boldsymbol{n}}\frac{\varepsilon_2}{\boldsymbol{p}^2}\right],$$

$$\xi = -\frac{1}{\nu' + \nu''}\left[\beta + \frac{\varepsilon_2}{(\boldsymbol{\Delta}\boldsymbol{n})}\right],$$

$S_{\pm}$ is the symmetrization or antisymmetrization operation:

$$S_+ f(\boldsymbol{e}) = f(\boldsymbol{e}) + f(-\boldsymbol{e})\ , \qquad S_- f(\boldsymbol{e}) = (1/\lambda)\left(f(\boldsymbol{e}) - f(-\boldsymbol{e})\right)\ .$$

These operations are necessary for the elimination of the ambiguity of the root: $\sqrt{E^2 - E_p^2}$.

Having introduced the functions Φ^r and Φ^a we reestablish the connection between E and $\boldsymbol{Q}$:

$$(1.30)\quad E^2 - \boldsymbol{Q}^2 = m_Q^2\ .$$

and substitute the expression (1.9)(see (1.12)) for

$$(1.31)\quad \boldsymbol{Q} = \boldsymbol{e}\sqrt{E^2 - E_p^2} + \eta\boldsymbol{p} + \xi\boldsymbol{\Delta}\ .$$

The functions Φ^r and Φ^a, thanks to the exponential multiplier are analytical in the upper and lower half-planes of the energy E respectively.

From (1.28) it follows that the functions Φ^r and Φ^a coincide on the region of the real axis $|E|<E_q$.

This means that there exists a unique function $\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho)$

$$\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \Delta, \varrho) = \begin{cases} \Phi^r(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho) & \text{when } \operatorname{Im} E > 0\,, \\ \Phi^a(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho) & \text{when } \operatorname{Im} E < 0\,, \end{cases} \tag{1.32}$$

which is analytical in the whole plane of the complex variable E except for the cuts

$$-\infty < \operatorname{Re} E < -E_q\,; \qquad E_q < \operatorname{Re} E < \infty\,, \qquad \operatorname{Im} E = 0\,. \tag{1.33}$$

Φ on the upper edge of the cuts is equal to Φ^r, and on the lower edge is equal to Φ^a.

Using the analytical properties of the function Φ, in the next paragraph we will obtain the dispersion relations for those cases in which the unobservable region is absent.

2. – Dispersion relations.

The function $\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho)$ when E tends to infinity increases of an order not higher than $n+12$.

Then to the function

$$\frac{\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho)}{(E - E_0)^{n+12}}\,, \tag{2.1}$$

where E_0 is the real parameter, $|E_0|<E_q$, it is possible to apply the Cauchy's integral theorem with the integration contour shown in Fig. 1.

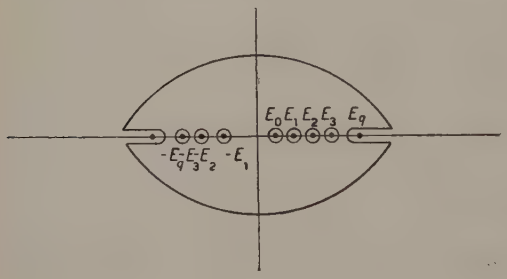

Fig. 1.

Letting the radius of the big circle go to infinity and that of the small semicircle to zero, we obtain:

$$\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho) = \tag{2.2}$$

$$= \frac{(E - E)^{n+12}}{2\pi i} \int_{\infty}^{-E_q} \frac{\Phi(E' + i0, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho) - \Phi(E' - i0, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho)}{(E' - E)(E' - E_0)^{n+12}}\, \mathrm{d}E' +$$

$$+ \frac{(E - E_0)^{n+12}}{2\pi i} \int_{E_q}^{\infty} \mathrm{d}E' \frac{\Phi(E' + i0, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho) - \Phi(E' - i0, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, \varrho)}{(E' - E)(E' - E_0)^{n+12}} + p_n(E)\,.$$

The boundary of the continuous spectrum E_q is determined from (1.18) so that the boundary of the spectrum satisfies the condition:

$$E_{ic}^{\min} \geqslant E_q \qquad (i = 1, 2, 3). \tag{2.3}$$

It is quite obvious, that condition (2.3) will be fulfilled if one can choose E_q equal to:

$$(\nu' + \nu'')E_q = -p_0 + \sqrt{(M+\mu)^2 - \boldsymbol{p}^2}\,. \tag{2.4}$$

The condition of absence of the unobserved region in dispersion relations has the following form:

$$E_q \geqslant E_p\,. \tag{2.5}$$

This condition sets some limit on the possible values of variables $\boldsymbol{p}$, $\boldsymbol{\Delta}$, ν'. Substituting the expression (1.15) and (2.4) into (2.5) we obtain, that is:

$$D = \sqrt{(M+\mu)^2 - \boldsymbol{p}^2} - p_0 \geqslant \frac{(\boldsymbol{p}^2 + m_\varrho^2)}{|\boldsymbol{p}|}\,, \tag{2.6}$$

then for any values of the angles between the vectors $\boldsymbol{p}$ and $\boldsymbol{\Delta}$, the corresponding values of the difference $(\nu' - \nu'')$, which lies between $(\nu' - \nu'')_1$ and $(\nu' - \nu'')_2$, when the unobservable region is absent, is:

$$(\nu' - \nu'')_{1,2} = \frac{\cos(\widehat{\boldsymbol{p}\boldsymbol{\Delta}})[(\boldsymbol{p}^2 - m_\varrho^2)/|\boldsymbol{p}|] \pm \sin(\widehat{\boldsymbol{p}\boldsymbol{\Delta}})\sqrt{D^2 - (\boldsymbol{p}^2 + m_\varrho^2)^2/\boldsymbol{p}^2}}{2m_\varrho}\,. \tag{2.7}$$

Notice that if

$$\sqrt{(M+\mu)^2} - p_0 > |\boldsymbol{p}|\,, \tag{2.8}$$

then there exists some interval of possible values of m_ϱ, at which the inequality (2.6) is satisfied. As we saw before by some restrictions on m_ϱ and p, the unobservable region is absent and the integration in (2.2) is carried out only for the observable energy region.

This is very important since the functions Φ^r and Φ^a are determined in the observable region and for $\varrho = 0$. In connection with this we may put ϱ equal to zero in the integrals (2.2).

And it is quite obvious, that the right hand side of the expression (2.2) will be an analytical function in the whole plane of the complex variable E, except for the cuts (1.33). Therefore, the function $\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, 0)$ will be analytical in the whole plane of the complex variable E, except for the

same cuts. But the function $\Phi(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}, 0)$ differs from the function:

$$S_{\pm}G(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta}) = \begin{cases} S_{\pm}T^{\mathrm{ret}}(E, \boldsymbol{e}, \boldsymbol{p}, \Delta) & \text{when} \quad \mathrm{Im}\, E > 0\,, \\ S_{\pm}T^{\mathrm{adv}}(E, \boldsymbol{e}, \boldsymbol{p}, \Delta) & \text{when} \quad \mathrm{Im}\, E < 0\,, \end{cases} \tag{2.9}$$

only by factors, which go to zero at the points

$$E = \pm E_1\,, \qquad E = \pm E_2\,, \qquad E = \pm E_3\,. \tag{2.10}$$

Therefore, the function $S_{\pm}G(E, \boldsymbol{e}, \boldsymbol{p}, \Delta)$ will be analytical in the whole E plane of the complex variable E with first degree poles at the points $E = \pm E_1$, $E = \pm E_2$, $E = \pm E_3$ and with cuts from $-\infty$ up to $-E_q$ and from E_q up to E_i. The E_i are solutions of the equation (1.16) when there is the connection (1.30). The function $S_{\pm}G(E, \boldsymbol{e}, \boldsymbol{p}, \Delta)$ has a degree of increase not higher than n. Therefore, to the function:

$$\frac{S_{\pm}G(E, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{\Delta})}{(E - E_0)^{n+1}}; \tag{2.11}$$

it is possible to apply Cauchy's theorem with the contour of integration shown in Fig. 1.

Letting the radius of the big circle go to infinity, and those of the small semi-circles to zero, we obtain:

$$S_{\pm}G(E, \boldsymbol{e}, \boldsymbol{p}, \Delta) = \frac{(E - E_0)^{n+1}}{2\pi i} \int\limits_{|E'| \geqslant E_q} \frac{S_{\pm}T^{\mathrm{ret}}(E') - S_{\pm}T^{\mathrm{adv}}(E')}{(E' - E)(E' - E_0)^{n+1}}\, \mathrm{d}E' + \tag{2.12}$$

$$+ \sum_{i=\alpha,\beta,\gamma} \left[\left(\frac{E_0 - E}{E_0 - E_i} \right)^{n+1} \frac{S_{\pm}R_i(\boldsymbol{\Delta}, \boldsymbol{p}, e)}{(E_i - E)} + \left(\frac{E_0 - E}{E_0 + E_i} \right)^{n+1} \frac{S_{\pm}\Omega_i(\boldsymbol{p}, \boldsymbol{\Delta}, \boldsymbol{e})}{(E_i + E)} + p_n(E) \right];$$

where

$$\begin{cases} R_\alpha = -\dfrac{1}{\nu}\left(1 - \dfrac{\nu E_3}{p_0}\right) V_q(p'', p') D_{-q';q''}(p, p'')\,, \\ R_\beta = \dfrac{1}{\nu''}\left(1 + \dfrac{\nu'' E_2}{p_0}\right) V_{q''}(p'', p') D_{q;q'}(p, p'')\,, \\ R_\gamma = \dfrac{1}{\nu'}\left(1 + \dfrac{\nu' E_1}{p_0}\right) V_{q'}(p'', p') D_{q;q''}(p, p'')\,, \\ \Omega_\alpha = -\dfrac{1}{\nu}\left(1 - \dfrac{\nu E_3}{p_0}\right) D_{-q',q''}(p'', p') V_q(p, p'')\,, \\ \Omega_\beta = \dfrac{1}{\nu''}\left(1 + \dfrac{\nu'' E_2}{p_0}\right) D_{q;q'}(p'', p') V_{q''}(p, p'')\,, \\ \Omega_\gamma = \dfrac{1}{\nu'}\left(1 + \dfrac{\nu' E_1}{p_0}\right) D_{q;q''}(p'', p') V_{q'}(p, p'')\,. \end{cases} \tag{2.13}$$

An inhomogeneous term in the expression (2.12) arises because one-nucleon poles E_i, lying in the region $|E| < E_q$ are present.

In the formulas (2.13) the operators $V_{q'}$, $V_{q''}$, V_q are vertex operators (with one-photon end and two-nucleon ends), and the operators $D_{q;q''}$, $D_{q';q'}$, etc., four-vertex operators (with two-photon ends and two-nucleon ends).

For a detailed discussion of the operators $V_{q'}$, $V_{q''}$, V_q we refer to paper [2].

In the expression (2.12) E takes only complex values. Using the limit relations:

$$\text{(2.14)} \qquad \begin{cases} \lim\limits_{\varepsilon \to 0} G(E + i\varepsilon) = T^{\mathrm{ret}}(E)\,, \\ \lim\limits_{\varepsilon \to} G(E - i\varepsilon) = T^{\mathrm{adv}}(E)\,, \end{cases}$$

let us pass to real E:

$$\text{(2.15)} \qquad S_{\pm} D_{\alpha,\omega}(E) = \frac{(E - E_0)^{n+1}}{\pi} P \int\limits_{|E| \geqslant E_q} \mathrm{d}E' \, \frac{S_{\pm} A_{\alpha,\omega}(E')}{(E' - E)(E' - E_0)^{n+1}} +$$

$$+ \sum_{i=\lambda,\beta,\gamma} \left[\left(\frac{E_0 - E}{E_0 - E_i} \right)^{n-1} \frac{S_{\pm} R_i(\boldsymbol{p}, \boldsymbol{\Delta}, \boldsymbol{e})}{(E_i - E)} + \left(\frac{E_0 - E}{E_0 - E_i} \right)^{n+1} \frac{S_{\pm} \Omega_i(\boldsymbol{p}, \boldsymbol{\Delta}, \boldsymbol{e})}{(E_i + E)} \right] + p_n(E)\,,$$

where $D_{\lambda,\omega}(E)$ and $A_{\lambda,\omega}(E)$ are the hermitian and antihermitian parts of the amplitude,

$$\text{(2.16)} \qquad \begin{cases} D_{\alpha,\omega}(E) = \tfrac{1}{2} \left(T^{\mathrm{ret}}_{\lambda,\omega}(E) + T^{\mathrm{adv}}_{\lambda,\omega}(E) \right), \\ A_{\alpha,\omega}(E) = (1/2i) \left(T^{\mathrm{ret}}_{\alpha,\omega}(E) - T^{\mathrm{adv}}_{\alpha,\omega}(E) \right). \end{cases}$$

Since in the interval (E_q, E_c)

$$E_c = \frac{M\mu + \frac{1}{2}\mu^2 - \boldsymbol{p}^2 - m_Q^2}{(\nu' + \nu'')\sqrt{\boldsymbol{p}^2 + M^2}}$$

the antihermitian part is equal to zero, then in the formula (2.15) the integration is essentially carried out from E_c.

The negative energy region in the dispersion relation (2.15) can be excluded using the symmetry properties of the antihermitian part,

$$A_{\alpha,\omega}(E) = - P_{ss'} A^*_{\alpha,\omega}(-E)\,,$$

$P_{ss'}$ is the operator of permutation of the spin of the nucleons.

* * *

In conclusion, the authors of the article express their deep gratitude to the Academician N. N. BOGOLJUBOV for his valuable advice and discussion.

RIASSUNTO (*)

Si motivano le relazioni di dispersione per l'effetto Compton doppio in caso di assenza della regione d'energia inosservabile.

(*) *Traduzione a cura della Redazione.*

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Einstein's Connections.

III - Degenerate Cases of Second Class.

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(ricevuto il 6 Giugno 1958)

Summary. — In a recent paper (3), I gave the solution of Einstein's first set of equations of unified field theory, by the method of repeated substitution, and obtained the conditions for the existence and uniqueness of the solution for the first two classes of $k_{\lambda\mu}$ and for all indices of inertia of $h_{\lambda\mu}$(*). The object of this paper is to study the solutions in the singular cases of the second class when these conditions are not satisfied. It may, however, be noted that Hlavatý (1) studied the singular solutions both in non-holonomic form and holonomic form. In this paper, the singular solutions will be obtained in other holonomic forms using the solution obtained by me (3) for non-degenerate case.

(*) The conditions are always satisfied in the third and fourth classes, provided the classes exist.

1. – Introduction.

In Einstein's unified field theory the basic real tensor $g_{\lambda\mu}$ is non-symmetric and may be split into its symmetric part $h_{\lambda\mu}$ and its skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} . \tag{1.1}$$

The basic connection $\Gamma^{\nu}_{\lambda\mu}$ is also non-symmetric and is given by

$$D_{\omega} g_{\lambda\mu} = 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} , \tag{1.2}$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} \Gamma^\nu_{[\lambda\mu]} = \tfrac{1}{2}(\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda}). \tag{1.3}$$

In the above $h_{\lambda\mu}$ is assumed to be of rank 4. It is said to be of index of inertia 0, 2 or 4 according as the signature of $h_{\lambda\mu}$ is $(++--)$, $(+++-)$ or $(++++)$ (*).

Denoting by $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{k}$ the determinants of $h_{\lambda\mu}$, $g_{\lambda\mu}$ and $k_{\lambda\mu}$ and taking $h^{\lambda\mu}$ as a tensor inverse to $h_{\lambda\mu}$ let us put

$$g \overset{\text{def}}{=} \frac{\mathfrak{g}}{\mathfrak{h}}, \tag{1.4}a$$

$$k \overset{\text{def}}{=} \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{1.4}b$$

$${}^{(0)}k_\lambda{}^\nu \overset{\text{def}}{=} \delta_\lambda{}^\nu, \quad {}^{(1)}k_\lambda{}^\nu \overset{\text{def}}{=} k_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu \overset{\text{def}}{=} {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu \qquad (p = 2, 3, \ldots), \tag{1.4}c$$

$$4K \overset{\text{def}}{=} k_{\alpha\beta}k_{\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} = k_{\alpha\beta}k^{\alpha\beta} = -{}^{(2)}k_\alpha{}^\alpha. \tag{1.4}d$$

We then have

$$g = 1 + 2K + k. \tag{1.5}$$

The tensor $g_{\lambda\mu}$ or $k_{\lambda\mu}$ is said to be

a) of the first class if $k \neq 0$,
b) of the second class if $K \neq 0$, $k = 0$,
c) of the third class if ${}^{(2)}k_\lambda{}^\nu \neq 0$, $K = k = 0$,
d) of the fourth class if ${}^{(2)}k_\lambda{}^\nu = 0$ (+), $k = 0$ (MISHRA (2)).

It has been shown by HLAVATÝ ((1), pp. 52-53) that if the system (1.2) admits a solution $\Gamma^\nu_{\lambda\mu}$ it must be of the form

$$\Gamma^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^\nu{}_{\lambda\mu}, \tag{1.6}$$

where $\left\{ {\nu \atop \lambda\mu} \right\}$ are the Christoffel symbols with respect to the tensor $h_{\alpha\beta}$ and

$$U^\nu{}_{\lambda\mu} \overset{\text{def}}{=} 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}. \tag{1.7}a$$

(*) This includes all equivalent combinations of $+$ $-$ signs also.
(+) This means that K also vanishes.
(1) V. HLAVATÝ: *Geometry of Einstein's Unified Field Theory* (Groningen, 1958).
(2) R. S. MISHRA: *Einstein's connections: Tensor* (to appear).

The tensors $S_{\lambda\mu}{}^{\nu}$ and $U^{\nu}_{\lambda\mu}$ are also related by

$$S_{\omega\mu\nu} = \tfrac{1}{2}K_{\omega\mu\nu} - 2U_{\alpha\nu[\mu}k_{\omega]}{}^{\alpha} \tag{1.7}b$$

where

$$K_{\omega\mu\nu} \overset{\text{def}}{=} \nabla_{\omega}k_{\nu\mu} + \nabla_{\mu}k_{\omega\nu} + \nabla_{\nu}k_{\omega\mu}\,, \tag{1.8}$$

the operator ∇_{μ} being the symbolic vector of the covariant derivative with respect to $\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$.

Eliminating $U_{\omega\mu\nu}$ from the equations (1.7) we obtain

$$S_{\omega\mu\nu} = B_{\omega\mu\nu} + k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}S_{\alpha\beta\nu} + k_{\omega}{}^{\alpha}k_{\nu}{}^{\gamma}S_{\alpha\mu\gamma} + k_{\mu}{}^{\beta}k_{\nu}{}^{\gamma}S_{\omega\beta\gamma} \tag{1.9}a$$

where

$$B_{\omega\mu\nu} = \tfrac{1}{2}(K_{\omega\mu\nu} + 3k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}K_{[\alpha\beta\nu]})\,, \tag{1.9}b$$

since

$$S_{[\omega\mu\nu]} = \tfrac{1}{2}K_{[\omega\mu\nu]}\,. \quad (*) \tag{1.9}c$$

Eliminating $S_{\omega\mu\nu}$ from the equations (1.7), we obtain

$$U_{\omega\mu\nu} = 2C_{\omega(\mu\nu)} + k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}U_{\alpha\beta\nu} + k_{\omega}{}^{\alpha}k_{\nu}{}^{\gamma}U_{\alpha\mu\gamma} + k_{\mu}{}^{\beta}k_{\nu}{}^{\gamma}U_{\omega\beta\gamma}\,, \tag{1.10}a$$

where

$$C_{\omega\mu\nu} = K_{\omega\mu\alpha}k_{\nu}{}^{\alpha} \tag{1.10}b$$

since

$$U_{(\alpha\beta\gamma)} = 0\,. \tag{1.10}c$$

For the second class, we have the relations (MISHRA [3])

$$^{(3)}k_{\omega}{}^{\alpha} = -2Kk_{\omega}{}^{\alpha} \tag{1.11}a$$

$$(^{(2)}k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta} + k_{\omega}{}^{\alpha}\,{}^{(2)}k_{\mu}{}^{\beta})S_{\alpha\beta\gamma} = 0\,, \tag{1.11}b$$

$$(^{(2)}k_{\omega}{}^{\alpha}\,{}^{(2)}k_{\mu}{}^{\beta} - 2Kk_{\omega}{}^{\alpha}k_{\mu}{}^{\beta})S_{\alpha\beta\gamma} = 0\,. \tag{1.11}c$$

(*) Here we use the notation

$$6\,T_{[\omega\mu\nu]} = (T_{\omega\mu\nu} + T_{\mu\nu\omega} + T_{\nu\omega\mu}) - (T_{\mu\omega\nu} + T_{\omega\nu\mu} + T_{\nu\mu\omega})$$
$$6\,T_{(\omega\mu\nu)} = (T_{\omega\mu\nu} + T_{\mu\nu\omega} + T_{\nu\omega\mu}) + (T_{\mu\omega\nu} + T_{\omega\nu\mu} + T_{\nu\mu\omega})\,.$$

(3) R. S. MISHRA: *Einstein's connections*, II: *Non-degenerate case*, in *Journ. Math. and Mech.* (to appear).

With the help of these relations the solutions of (1.9) and (1.10) in the second class have been found to be

$$(1.12)\qquad (S_{\omega\mu\nu}-B_{\omega\mu\nu})(1-4K^2)=[2k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}{}^{(2)}k_{\nu}{}^{\gamma}+{}^{(2)}k_{\omega}{}^{\alpha}{}^{(2)}k_{\nu}{}^{\gamma}\delta_{\mu}{}^{\beta}+$$
$$+\delta_{\omega}{}^{\alpha}{}^{(2)}k_{\mu}{}^{\beta}{}^{(2)}k_{\nu}{}^{\gamma}-k_{\omega}{}^{\alpha}\delta_{\mu}{}^{\beta}k_{\nu}{}^{\gamma}-\delta_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}k_{\nu}{}^{\gamma}-(1-2K)k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}\delta_{\nu}{}^{\gamma}]B_{\alpha\beta\gamma}$$

and

$$(1.13)\qquad (U_{\omega\mu\nu}-2C_{\omega(\mu\nu)})(1-4K^2)=2[2k_{\omega}{}^{\alpha}k_{(\mu}{}^{\beta}{}^{(2)}k_{\nu)}{}^{\gamma}+{}^{(2)}k_{\omega}{}^{\alpha}{}^{(2)}k_{(\nu}{}^{\gamma}\delta_{\mu)}{}^{\beta}+$$
$$+\delta_{\omega}{}^{\alpha}{}^{(2)}k_{(\mu}{}^{\beta}{}^{(2)}k_{\nu)}{}^{\gamma}-k_{\omega}{}^{\alpha}\delta_{(\mu}{}^{\beta}k_{\nu)}{}^{\gamma}-\delta_{\omega}{}^{\alpha}k_{(\mu}{}^{\beta}k_{\nu)}{}^{\gamma}-(1-2K)k_{\omega}{}^{\alpha}k_{(\mu}{}^{\beta}\delta_{\nu)}{}^{\gamma}]\,C_{\alpha\beta\gamma}\,.$$

For the first two classes there are exactly four linearly independent vectors $\underset{i}{a}{}^{\nu}$ satisfying (Hlavatý [1], pp. 14, 16)

$$(1.14)a\qquad \underset{x}{\lambda}\,\underset{x}{a}{}^{\nu}=k_{\lambda}{}^{\nu}\,\underset{x}{a}{}^{\lambda}\,,\qquad b)\qquad h_{\lambda\mu}\,\underset{x}{a}{}^{\lambda}\,\underset{x}{a}{}^{\mu}=0\ \ (*)\,,$$

where $\underset{x}{\lambda}$ are solutions of

$$(1.15)\qquad \lambda^4+2K\lambda^2+k=0\,.$$

For the first two classes we have

$$(1.16)a\qquad \underset{1}{\lambda}=-\underset{2}{\lambda}=i\sqrt{\sqrt{D}+K}\neq 0\,,$$

$$(1.16)b\qquad \underset{3}{\lambda}=-\underset{4}{\lambda}=\sqrt{\sqrt{D}-K}\neq 0\,,$$

where

$$(1.16)c\qquad D=K^2-k$$

and for the second class we have

$$(1.17)a\qquad \underset{1}{\lambda}=-\underset{2}{\lambda}=\sqrt{-2K}\,,\qquad \text{for}\quad K>0,$$

$$(1.17)b\qquad \underset{3}{\lambda}=-\underset{4}{\lambda}=0\,,$$

or

$$(1.18)a\qquad \underset{1}{\lambda}=-\underset{2}{\lambda}=0\,,\qquad \text{for}\quad K<0.$$

$$(1.18)b\qquad \underset{3}{\lambda}=-\underset{4}{\lambda}=\sqrt{-2K}\,,$$

(*) The indices x, y, z, t do not follow the summation convention. The summation for these indices is denoted by $\sum$.

If the non-holonomic components of $h_{\lambda\mu}$, $h^{\lambda\mu}$ and ${}^{(p)}k_{\lambda}^{\ \nu}$ in the first two classes are given by

$$(1.19)a \qquad (\!(h_{ij})\!) = (\!(h^{ij})\!) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$(1.19)b \qquad {}^{(p)}k_{x}^{\ i} = (\underset{x}{\lambda})^{p}\delta_{x}^{\ i}$$

then the solutions of (1.9) and (1.10) in non-holonomic frame are given by (HLAVATÝ [1], pp. 79, 80)

$$(1.20)a \qquad \underset{xyz}{\lambda} S_{xyz} = \tfrac{1}{2}(K_{xyz} + 3\underset{x}{\lambda}\underset{y}{\lambda} K_{[xyz]})$$

with

$$(1.20)b \qquad S_{[xyz]} = \tfrac{1}{2} K_{[xyz]}$$

and

$$(1.21)a \qquad \underset{xyz}{\lambda} U_{xyz} = \tfrac{1}{2}(K_{xyz}\underset{z}{\lambda} + K_{xzy}\underset{y}{\lambda})$$

with

$$(1.21)b \qquad U_{(xyz)} = 0\,.$$

From (1.12), (1.13) or from (1.20), (1.21) and (1.17) or (1.18), it is clear that the necessary and sufficient conditions for the existence and uniqueness of the solution $S_{\omega\mu\nu}(U_{\omega\mu\nu})$ in the second class are

$$(1.22)a \qquad g(g-2) \neq 0$$

equivalent to

$$(1.22)b \qquad 1 - 4K^{2} \neq 0\,.$$

When the index of inertia of $k_{\lambda\mu}$ is 4, K is always positive. Hence the condition (1.22) in this case reduces to (MISHRA [2])

$$(1.23) \qquad g - 2 = 2K - 1 \neq 0\,.$$

HLAVATÝ ([1], pp. 114-127) found the necessary and the sufficient conditions for the existence of at least one solution $S_{\omega\mu\nu}$ of (1.19) when

$$g = 0$$

or

$$g = 2\,.$$

For the degenerate case

$$g = 0$$

we have

(1.24)a $$\alpha \overset{\text{def}}{=} \lambda_1 = -\lambda_2 = 0$$

(1.24)b $$\beta \overset{\text{def}}{=} \lambda_3 = -\lambda_4 = 1$$

and

(1.25)a $$\lambda_{34i} = 0$$

(1.25)b $$\lambda_{12i} = \lambda_{aaf} = 1$$

(1.25)c $$\lambda_{aff} = 2 \quad (*).$$

The necessary and sufficient conditions for the existence of at least one solution $S_{\omega\mu\nu}$ in this case reduce to (HLAVATÝ [1], pp. 121, 122)

(1.26) $$K_{ief} = K_{i(ef)} = 0$$

and when these conditions are satisfied the tensor $S_{\omega\mu\nu}$ has the following non-holonomic components:

(1.27)a The components S_{eff} are arbitrary,

(1.27)b The components S_{efa}, S_{fae} are arbitrary except for the single condition (+) $\frac{1}{2}K_{efa} = 3S_{[efa]}$

(1.27)c $$2S_{fab} = K_{fab}$$

(1.27)d $$2S_{abf} = K_{abf}$$

(1.27)e $$2S_{eae} = \tfrac{1}{2}K_{eae}$$

(1.27)f $$S_{abc} = 0\,.$$

For the degenerate case

$$g = 2$$

we have (HLAVATÝ [1], pp. 123, 126)

(1.28)a $$\alpha \overset{\text{def}}{=} \lambda_1 = -\lambda_2 = i$$

(1.28)b $$\beta \overset{\text{def}}{=} \lambda_3 = -\lambda_4 = 0$$

(*) In what follows a, b, c, d, will run from 1 to 2 and e, f, g, h, will run from 3 to 4. Also $a \neq b$, $e \neq f$.

(+) This condition has been put not to violate the condition (1.26).

and

(1.29)a $$\underset{12i}{\lambda} = 2\,,$$

(1.29)b $$\underset{34i}{\lambda} = \underset{afg}{\lambda} = 1\,,$$

(1.29)c $$\underset{aaf}{\lambda} = 0\,.$$

A necessary and sufficient condition for the system (1.9) to admit at least one solution $S_{\omega\mu\nu}$ is

(1.30) $$K_{faa} = 0\,.$$

When this condition is satisfied, the tensor $S_{\omega\mu\nu}$ has the following non-holonomic components:

(1.31)a the four components S_{faa} are arbitrary

(1.31)b $$2S_{efa} = K_{efa}\,,$$

(1.31)c $$2S_{abf} = \tfrac{1}{2}(K_{abf} + 3K_{[abf]})\,,$$

(1.31)d $$2S_{afb} = \tfrac{1}{2}K_{afb}\,,$$

(1.31)e $$2S_{afg} = K_{afg}\,,$$

(1.31)f $$S_{efg} = S_{abc} = 0\,.$$

In both cases, HLAVATÝ found the geometrical interpretation of (1.26) and (1.30) independently of the non-holonomic frame.

In the following we shall also make use of the vectors $\overset{i}{a}_\lambda$ inverse to $\overset{\lambda}{a_i}$, such that

(1.32)a $$\overset{i}{a}_\lambda \overset{\lambda}{a_j} = \delta^i_j\,,$$

(1.32)b $$\overset{i}{a}_\lambda \overset{\mu}{a_i} = \delta^\mu_\lambda\,.$$

If we put

(1.33)a $$\overset{xy}{P}_{\lambda\mu} \overset{\text{def}}{=} 2\overset{x}{a}_{(\lambda}\overset{y}{a}_{\mu)}\,,$$

(1.33)b $$\underset{xy}{Q}^{\lambda\mu} \overset{\text{def}}{=} 2\overset{\lambda}{a}_{[x}\overset{\mu}{a}_{y]}\,,$$

then it has been shown that (HLAVATÝ (1), p. 26) the tensors P and Q can be constructed without the use of basic vectors. Their values in terms of $g_{\lambda\mu}$ are given in HLAVATÝ ((1), p. 26).

2. – Singular solutions. $g = 0$.

The conditions of HLAVATY ([1], pp. 120-127) for the existence of at least one solution $S_{\omega\mu\nu}$ of (1.9) when $g = 0$ or $g = 2$ are in both holonomic and non-holonomic forms for the index of inertia 2. I shall now find the conditions for all indices of inertia in other holonomic forms. The solutions $S_{\omega\mu\nu}$ and $U_{\omega\mu\nu}$ for these degenerate cases are also obtained in other holonomic forms.

If we put

$$(2.1)a \qquad \overset{xyz}{T} \overset{\text{def}}{=} \overset{xyz}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} T_{\alpha\beta\gamma} \overset{\text{def}}{=} {}^{(x)}k_{\omega}{}^{\alpha}\, {}^{(y)}k_{\mu}{}^{\beta}\, {}^{(z)}k_{\nu}{}^{\gamma}\, T_{\alpha\beta\gamma},$$

$$(2.1)b \qquad \overset{000}{T} \overset{\text{def}}{=} T_{\omega\mu\nu}$$

and denote

$$(2.2)a \qquad 6\,\overset{(xyz)}{T} \overset{\text{def}}{=} (\overset{xyz}{T} + \overset{yzx}{T} + \overset{zxy}{T}) + (\overset{yxz}{T} + \overset{zyx}{T} + \overset{xzy}{T})$$

$$(2.2)b \qquad 2\,\overset{(xy)z}{T} \overset{\text{def}}{=} \overset{xyz}{T} + \overset{yxz}{T},$$

where $T_{\alpha\beta\gamma}$ is any tensor, then the equations (1.11) when $2K = -1$ are replaced by

$$(2.3)a \qquad {}^{(3)}k_{\omega}{}^{\alpha} = k_{\omega}{}^{\alpha},$$

$$(2.3)b \qquad \overset{(12)r}{S} = 0.$$

$$(2.3)c \qquad \overset{22r}{S} + \overset{11r}{S} = 0,$$

If further

$$\overset{pqr}{T} \times \overset{xyz}{A} \overset{\text{def}}{=} \overset{pqr}{T}_{\alpha\beta\gamma} \overset{xyz}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}$$

then we have:

Theorem (2.1). *In the second class when* $g = 0$

$$(2.4)a \qquad 3\,\overset{00c}{S} \times \overset{(110)}{A} = 2\,\overset{(10)1}{S} + \overset{110}{S}$$

$$(2.4)b \qquad 3\,\overset{(10)1}{S} \times \overset{(110)}{A} = \overset{112}{S} + \overset{(20)2}{S}$$

$$(2.4)c \qquad 3\,\overset{(20)2}{S} \times \overset{(110)}{A} = \overset{112}{S} + \overset{(10)1}{S}$$

$$(2.4)d \qquad 3\,\overset{110}{S} \times \overset{(110)}{A} = -\overset{110}{S}$$

$$(2.4)e \qquad 3\,\overset{112}{S} \times \overset{(110)}{A} = -\overset{112}{S}.$$

Proof. The equations (2.4) follow by computation and use of the equations (2.3) (*)

Theorem (2.2). *When* $g = 0$, *in the second class, the equations* (1.9) *induce the equations*

$$B = S + 2\overset{(10)1}{S} + \overset{110}{S} = S + 3\overset{(110)}{S} \tag{2.5}a$$

$$\overset{(10)1}{B} = \overset{(10)1}{S} + \overset{112}{S} + \overset{(20)2}{S} \tag{2.5}b$$

$$\overset{(20)2}{B} = \overset{(10)1}{S} + \overset{112}{S} + \overset{(20)2}{S} \tag{2.5}c$$

$$\overset{110}{B} = \overset{112}{B} = 0\,. \tag{2.5}d$$

Proof. The equation (1.9)a, written with the help of (2.1) and (2.2) assumes the form (2.5)a.

Applying (+) the tensors $\overset{(10)1}{A}$, $\overset{(20)2}{A}$, $\overset{110}{A}$, $\overset{112}{A}$ to B and using (2.4), we obtain (2.5).

Note. A simple inspection of the equations (2.5) shows that the equations (2.5)b, c are either inconsistent or, if consistent, they reduce to one equation. Therefore, if the system (2.5)a-c is consistent, it contains only two independent equations, and therefore does not uniquely define the four unknowns S, $\overset{(10)1}{S}$, $\overset{110}{S}$, $\overset{112}{S} + \overset{(20)2}{S}$.

We, therefore, have the following theorem:

Theorem (2.3). *A necessary and sufficient condition that the equations* (2.5) *admit at least one solution* S *is*

$$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})B_{\alpha\beta\gamma} = 0 \tag{2.6}a$$

equivalent to

$$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})K_{\alpha\beta\gamma} = 0 \tag{2.6}b$$

which in non-holonomic form reduces to (×)

$$K_{ief} = K_{i(ef)} = 0\,. \tag{2.6}c$$

(*) For details of computation, the reader is referred to Mishra ([3]).

(+) Applying $\overset{pqr}{A}$ to B means

$$\overset{pqr}{A} \times B = \overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} B_{\alpha\beta\gamma}\,.$$

(×) This is Hlavatý's condition ([1], p. 121).

Proof. Eliminating $\overset{\nu\varrho\tau}{S}$ from (2.5)b, c (assuming these equations as consistent), we obtain (2.6)a.

Writing

$$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})\,\overset{110}{A}{}^{\varrho\sigma\delta}_{\alpha\beta\gamma} K_{[\varrho\sigma\delta]}$$

in non-holonomic form and using (1.19)b and (1.24), we find that this expression vanishes.

Hence substituting the value of $B_{\gamma\beta\gamma}$ from (1.9)b in (2.6)a, we get (2.6)b.

Writing (2.6) in non-holonomic form and using (1.19)b and (1.24), we obtain (2.6)c.

Remark. The equations (2.5)d are

$$\overset{110}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} B_{\alpha\beta\gamma} = 0$$

$$\overset{112}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} B_{\alpha\beta\gamma} = 0$$

which in consequence of (1.9)b are equivalent to

(2.7)a $$\overset{110}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}(K_{\alpha\beta\gamma} + 3k_{\alpha}^{\varrho} k_{\beta}^{\sigma} K_{[\varrho\sigma\gamma]}) = 0$$

and

(2.7)b $$\overset{112}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma} = 0\,.$$

Writing these equations in non-holonomic forms and using (1.19)b and (1.24), we obtain respectively

(2.8)a $$K_{i[ef]} = 0\,,$$

(2.8)b $$K_{g[ef]} = 0\,.$$

Thus we see that (2.8) are contained in (2.6)c. Hence (2.5)d or (2.7) are the conditions for the existence of at least one solution S of (2.5), contained in (2.6)a or (2.6)b.

Theorem (2.4). *A necessary and sufficient condition for the existence of at least one solution $S_{\omega\mu\nu}$ of* (1.9) *is* (2.6).

Proof. The statement follows from Theorem (2.3) and the fact that the solution $S_{\omega\mu\nu}$ of (1.9) is to be obtained from the equations (2.5).

With the help of the condition (1.26), HLAVATÝ has shown that ([1], p. 122):

A necessary and sufficient condition that (1.9) *admit at least one solution $S_{\omega\mu\nu}$ when $g = 0 = k$ is that the basic vectors $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$* are X_2-forming.

With the help of this result we can express the condition (2.6) *in* another form, and we have the following theorem:

Theorem (2.5). *The necessary and sufficient condition for the existence of at least one solution* $S_{\omega\mu\nu}$ *of* (1.9) *is also expressible in the form*

$$\underset{34}{Q}{}^{\lambda\mu}\,\partial_\mu \overset{12}{P}_{\lambda\mu} = 0\,. \tag{2.9}$$

Proof. Since the basic vectors $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$ are X_2-forming, we have

$$\underset{[3}{a}{}^{\lambda}\,\underset{4]}{a}{}^{\mu}\nabla_\mu \overset{a}{a}_\lambda = 0\,. \tag{2.10}$$

Now in consequence of (1.33)

$$\underset{34}{Q}{}^{\lambda\mu}\,\nabla_\mu \overset{12}{P}_{\lambda\nu} = 4\underset{3}{a}{}^{[\lambda}\underset{4}{a}{}^{\mu]}\nabla_\mu(\overset{1}{a}_{(\lambda}\overset{3}{a}_{\nu)}) =$$

$$= 2\underset{3}{a}{}^{[\lambda}\underset{4}{a}{}^{\mu]}(\overset{1}{a}_\lambda\nabla_\gamma\overset{3}{a}_\nu + \overset{3}{a}_\lambda\nabla_\mu\overset{1}{a}_\nu) + 2\underset{3}{a}{}^{[\lambda}\underset{4}{a}{}^{\mu]}(\overset{3}{a}_\nu\nabla_\mu\overset{1}{a}_\lambda + \overset{1}{a}_\nu\nabla_\mu\overset{3}{a}_\lambda)\,.$$

The expression within the first parenthesis on the right hand side of the above equation vanishes in consequence of (1.32) and the expression within the second parenthesis vanishes in consequence of (2.10).

Hence we have

$$\underset{34}{Q}{}^{\lambda\mu}\,\nabla_\mu \overset{12}{P}_{\lambda\nu} = \underset{31}{Q}{}^{\lambda\mu}\left(\partial_\mu \overset{12}{P}_{\lambda\nu} - \overset{12}{P}_{\alpha\nu}\left\{\begin{matrix}\alpha\\ \lambda\mu\end{matrix}\right\} - \overset{12}{P}_{\lambda\alpha}\left\{\begin{matrix}\alpha\\ \nu\mu\end{matrix}\right\}\right) = 0\,.$$

But $\underset{34}{Q}{}^{\lambda\mu}\left\{\begin{matrix}\alpha\\ \mu\lambda\end{matrix}\right\}$ vanishes since $\underset{31}{Q}{}^{\lambda\mu}$ is skew-symmetric in $\lambda\mu$ and $\underset{34}{Q}{}^{\lambda\mu}\overset{12}{P}_{\lambda\nu}$ vanishes in consequences of (1.32) and (1.33). Hence we have (2.9).

Theorem (2.6). *A necessary and sufficient condition that the basic vectors* $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$ *be* X_2*-forming when* $g = 0 = k$, *is* (2.6) *or* (2.9).

Proof. The proof is obvious from Hlavatý's result given at the end of Theorem (2.4) and from Theorem (2.5.).

Theorem (2.7). *When the condition* (2.6) *or* (2.9) *is satisfied the solution* $S_{\omega\mu\nu}$ *of* (1.9) *in the second class when* $g = 0$ *is*

$$S_{\omega\mu\nu} = \tfrac{1}{2}(K_{\omega\mu\nu} + 3k_\omega{}^\alpha k_\mu{}^\beta K_{[\alpha\beta\nu]}) + D_{\omega\mu\nu}\,, \tag{2.11}a$$

where $D_{\omega\mu\nu}$ *is a tensor skew-symmetric in the first two indices, such that in non-holonomic frame*

(2.11)*b* *the eight components* D_{efi}, D_{aef} $(e \neq f)$ *are arbitrary,*

$$D_{aee} = -\tfrac{1}{4}K_{aee} \qquad (e \neq f) \tag{2.11}c$$

(2.11)*d* *the remaining* $D_{ijk} = 0\,.$

Proof. The solution $S_{\omega\mu\nu}$ in the second class is given by ((1.12)) when $g(g-2) \neq 0$. When $g=0$, (1.9) gives at least one solution if (2.6), ((2.9)) is satisfied. Putting $2K=-1$ and substituting (2.6) in (1.12), we find that the solution is of the form (2.11)*a*.

Alternately we can prove this in the following way:

When (2.6)*a* is satisfied, the equations (2.5)*b*, *c* are consistent. Therefore $3\overset{(110)}{S}$ may be taken arbitrary. Hence from (2.5)*a*, we have (2.11)*a*.

When (2.6) is satisfied and $g=0$, the non-holonomic components S_{xyz} of $S_{\omega\mu\nu}$ in the second class are given by (1.27). Substituting these values of S_{ijk} in the non-holonomic form of (2.11)*a* and using (1.19)*b* and (1.24), we obtain (2.11)*b-d*.

Theorem (2.8). *A necessary and sufficient condition that the solution* (2.11)*a also satisfies the condition*

$$S_\lambda \overset{\text{def}}{=} S_{\lambda x}{}^{x} = 0 \tag{2.12}$$

is

$$K_\omega = 2k_\omega{}^\alpha k^{\nu\beta}\nabla_\beta k_{\nu\alpha} - 2D_\omega\,, \tag{2.13}a$$

where

$$K_\omega \overset{\text{def}}{=} K_{\omega\mu\nu}h^{\mu\nu} = 2\nabla_\alpha k_\omega{}^\alpha\,, \tag{2.13}b$$

$$D_\omega \overset{\text{def}}{=} D_{\omega\mu\nu}h^{\mu\nu}\,. \tag{2.13}c$$

In this case the non-holonomic components of $D_{\omega\mu\nu}$ *are given by*

$$D_{efg} = -K_{g(ab)}\,, \tag{2.14}a$$

D_{aef} *are arbitrary, but* $D_{a(ef)}=0$, (2.14)*b*

the remaining components are the same as in Theorem (2.7). (2.14)*c*

Proof. Multiplying (2.11)*a* by $h^{\mu\nu}$ and putting $S_\omega = 0$, we get (2.13)*a*, in consequence of (2.13)*b*, *c*.

Writing (2.13)*a* in non-holonomic form and using (1.19)*b* and (1.24), we obtain

$$K_{i(ab)} + 2D_{i(ab)} + 2D_{i(ef)} = 0\,. \tag{2.15}$$

But

$$K_{c(ab)} = 0\,.$$

Hence using (2.11)*b-d* in (2.15), we obtain (2.14).

Remark. Using (2.14) in non-holonomic form of (2.11)*a*, we find that in this case

a) the components S_{eff} are given by

$$S_{eff} = K_{f(ab)}\,, \qquad a \neq b,\ e \neq f;$$

b) components S_{aef} are arbitrary, while

$$S_{a(ef)} = 0\,;$$

c) the remaining components S_{ijk} are given by (1.27).

This result has already been proved by HLAVATÝ ([1], p. 123) by another method.

We will now consider the tensor $U_{\omega\mu\nu}$. In the second class when $g(g-2) \neq 0$, it is given by (1.13).

Theorem (2.9). *The necessary and sufficient condition for the existence of at least one solution* $U_{\omega\mu\nu}$ *of* (1.10) *in the second class when* $g = 0$ *is* (2.6).

Proof. The statement is obvious from Theorem (2.4) and (1.7)*a*.

Theorem (2.10). *When the condition* (2.6)*a is satisfied, the solution* $U_{\omega\mu\nu}$ *of* (1.10) *has non-holonomic components (symmetric in the last two indices) in the form*

(2.16)*a* $\qquad U_{efi},\ U_{ief}$ *are arbitrary*

(2.16)*b* $\qquad U_{abe} = \tfrac{1}{2}\lambda_e K_{abe}$

(2.16)*c* $\qquad U_{aee} = \tfrac{1}{2}\lambda_e K_{aee} \qquad a \neq b,\ e \neq f$

(2.16)*d* $\qquad U_{eae} = \tfrac{1}{4}\lambda_e K_{eae}$

(2.16)*e* *the remaining* $\qquad U_{bcd},\ U_{aae},\ U_{fbc},\ U_{eee} = 0\,.$

Proof. Writing (1.7)*a* in non-holonomic form and using (1.27), we obtain (2.16), the non-holonomic components U_{ijk} of $U_{\omega\mu\nu}$.

The non-holonomic components U_{ijk} can also be obtained directly from the equation (1.21)*a* by substituting from (1.24) and (1.25),.

Theorem (2.11). *When the condition* (2.6)*a is satisfied, the solution* $U_{\omega\mu\nu}$ *of* (1.10) *is of the form*

(2.17) $$U_{\omega\mu\nu} = K_{\omega\mu\alpha} k_\nu^{\ \alpha} + K_{\omega\nu\alpha} k_\mu^{\ \alpha} + E_{\omega\mu\nu}\,,$$

where $E_{\omega\mu\nu}$ is a tensor symmetric in the last two indices, The non-holonomic components E_{ijk} of the tensor $E_{\omega\mu\nu}$ are symmetric in the last two indices and are given by

(2.18)a *the ten components* E_{efi}, E_{aef} *are arbitrary*

(2.18)b $$E_{abe} = -\tfrac{1}{2}\lambda_e K_{abe}$$

(2.18)c $$E_{aee} = -\tfrac{3}{2}\lambda_e K_{aee}$$

(2.18)d $$E_{eae} = -\tfrac{3}{4}\lambda_e K_{eae}$$

(2.18)e $$E_{bcd} = E_{aae} = E_{fbc} = E_{eee} = 0\,.$$

Proof. Substituting the condition (2.6)a in (1.13), and putting $g = 0$, we find that the solution must be of the form (2.17).

Writing (2.17) in non-holonomic form and substituting the values of λ_i and U_{ijk} from (1.24) and (2.16) respectively and using (2.6)c, we get (2.18).

3. – Singular solutions. $g = 2$.

We will now consider the second degenerate case of the second class which is characterized by the conditions

$$K \neq 0\,, \qquad k = 0$$

and

$$2K = 1\,.$$

The equations (1.11) in this case assume the forms

(3.1)a $$^{(3)}k_{\omega}^{\ \mu} = -\,k_{\omega}^{\ \mu}$$

(3.1)b $$\overset{(12)r}{S} = 0$$

(3.1)c $$\overset{22r}{S} = \overset{11r}{S}\,.$$

Theorem (3.1). *In the second class when $g = 2$ we have*

(3.2)a $$3\overset{000}{S} \times \overset{(110)}{A} = 2\overset{(10)1}{S} + \overset{110}{S} = 3\overset{(11)}{S}$$

(3.2)b $$3\overset{(10)1}{S} \times \overset{(110)}{A} = \overset{112}{S} + \overset{(20)2}{S}$$

(3.2)c $$3\overset{(20)2}{S} \times \overset{(110)}{A} = -\overset{112}{S} + \overset{(10)1}{S}$$

(3.2)d $$3\overset{110}{S} \times \overset{(110)}{A} = \overset{110}{S}$$

(3.2)e $$3\overset{112}{S} \times \overset{(110)}{A} = \overset{112}{S}\,.$$

Theorem (3.2). *In the second class, when* $g = 2$ *the equations* (1.9) *induce the equations*

(3.3)*a* $$B = S + 2\overset{(10)1}{S} + \overset{110}{S}$$

(3.3)*b* $$\overset{(10)1}{B} = \overset{(10)1}{S} + \overset{112}{S} + \overset{(20)2}{S}$$

(3.3)*c* $$\overset{(20)2}{B} = \overset{(10)1}{S} - \overset{112}{S} + \overset{(20)2}{S}$$

(3.3)*d* $$\overset{112}{B} = 2\overset{112}{S}$$

(3.3)*e* $$\overset{110}{B} = 2\overset{110}{S}\,.$$

Proof. The proof of these theorems follows the pattern of the corresponding Theorems (2.1) and (2.2).

Note. A simple inspection of the equations (3.3) shows that the equations (3.3)*b-d* are either inconsistent or, if consistent, they reduce to only two equations, under certain condition. Therefore if the system (3.3) is consistent, it contains only four independent equations and, therefore, does not uniquely define the five unknown quantities S, $\overset{(10)1}{S}$, $\overset{110}{S}$, $\overset{112}{S}$, $\overset{(20)2}{S}$.

Theorem (3.3). *A necessary and sufficient condition for the existence of at least one solution* $S_{\omega\mu\nu}$ *of* (1.9) *is*

(3.4)*a* $$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{112}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})B_{\alpha\beta\gamma} = 0$$

equivalent to

(3.4)*b* $$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{112}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})K_{\alpha\beta\gamma} = 0\,.$$

In non-holonomic form the condition (3.4) *is equivalent to Hlavatý's* ([1], p. 124) *condition* (1.30).

Proof. Considering the equations (3.3)*b-d* as consistent and eliminating $\overset{112}{S}$, $\overset{(20)2}{S} + \overset{(10)1}{S}$ from these we get (3.4)*a*.

Writing

$$(\overset{(10)1}{A}{}^{\alpha\beta\gamma]}_{\omega\mu\nu} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu} - \overset{112}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu})\overset{110}{A}{}^{\theta\varphi\psi}_{\alpha\beta\gamma}K_{[\theta\varphi\psi]}$$

in non-holonomic form and using (1.19)*b* and (1.28), we find that the expression vanishes. Hence (3.4)*a* reduces to (3.4)*b*.

Since the solution $S_{\omega\mu\nu}$ of (1.19) is to be obtained from the equations (3.3), we have (3.4) as the necessary and sufficient condition for the existence of at least one solution $S_{\omega\mu\nu}$ of (1.9).

Writing (3.4)b in non-holonomic form and using (1.19)b, we have

$$\lambda_z(\lambda_x+\lambda_y)[1-\lambda_z(\lambda_x+\lambda_y)]K_{xyz}=0\,.$$

Using (1.28) we find that the coefficient of K_{xyz} is not equal to zero, only when $xyz=faa$ or $xyz=afa$. Hence we have the last part of the theorem.

Remark I. It is easy to see that (3.4) is necessary and sufficient not only for the existence of at least one solution S but also for the existence of at least one solution of any other unknown $\overset{(10)1}{S}$, $\overset{110}{S}$, $\overset{112}{S}$, $\overset{(20)2}{S}$.

Remark II. With the help of the condition (1.30), HLAVATÝ has shown that ([1], p. 124)):

A necessary and sufficient condition for the system (1.9) *to admit at least one solution is that the trajectories of* a^ν_α *be geodesic lines.*

Hence we have the following theorem:

Theorem (3.4). *A necessary and sufficient condition for the trajectories of* a^ν_α *to be geodesic lines, when* $k=0$, $g=2$ *is* (3.4).

Proof. The proof is obvious.

Theorem (3.5). *When the condition* (3.4) *is satisfied, the solution* $S_{\omega\mu\nu}$ *of* (1.9) *is of the form*

$$S_{\omega\mu\nu}=B_{\omega\mu\nu}-\tfrac{1}{2}k_\omega{}^\alpha k_\mu{}^\beta B_{\alpha\beta\nu}+F_{\omega\mu\nu} \tag{3.5a}$$

equivalent to

$$S_{\omega\mu\nu}=\tfrac{1}{2}(K_{\omega\mu\nu}+3k_\omega{}^\lambda k_\mu{}^\beta K_{[\lambda\beta\nu]})-\tfrac{1}{4}(k_\omega{}^\lambda k_\mu{}^\beta K_{\alpha\beta\nu}+3\,{}^{(2)}k_\omega{}^{\alpha\,(2)}k_\mu{}^\beta K_{[\lambda\beta\nu]})+F_{\omega\mu\nu}, \tag{3.5b}$$

where $F_{\omega\mu\nu}$ *is a tensor, skew-symmetric in the first two indices, such that in non-holonomic form*

$$\text{the four components}\quad F_{faa}\quad \text{are arbitrary} \tag{3.6a}$$

$$F_{afb}=-\tfrac{1}{4}K_{afb}\qquad a\neq b, \tag{3.6b}$$

$$\text{the remaining}\quad F_{ijk}=0\,. \tag{3.6c}$$

Proof. The solution in the second class is given by (1.12), when $g(g-2)\neq 0$. When $g=2$, (1.9) gives at least one solution if (3.4) is satisfied. Putting $2K=1$, and substituting (3.4)a, we find that the solution $S_{\omega\mu\nu}$ must be of the form (3.5)a.

Writing the value of $B_{\omega\mu\nu}$ from (1.9)b in (3.5)a we obtain (3.5)b.

The non-holonomic components S_{ijk} of $S_{\omega\mu\nu}$ when $g=2$, are given by (1.31).

Substituting these values of S_{ijk} in the non-holonomic form

$$S_{xyz} + \tfrac{1}{4}(\lambda_x \lambda_y - 2)(K_{xyz} + 3\lambda_x \lambda_y K_{[xyz]}) = F_{xyz} \tag{3.7}$$

of (3.5)*b*, we get (3.6)

HLAVATÝ ([1], p. 126) has found a necessary and sufficient condition for a solution in the second class when $g = 2$, to satisfy also (2.12), *viz.*

$$S_\lambda = 0 .$$

I will now find the condition in another form.

Theorem (3.6). *A necessary and sufficient condition that the solution* (3.5) *also satisfies the condition* (2.12) *is*

$$K_\omega + 2F_\omega = 0, \tag{3.8}a$$

where

$$F_\omega \overset{\text{def}}{=} F_{\omega\mu\nu} h^{\mu\nu} . \tag{3.8}b$$

When this condition is satisfied, we get Hlavatý's result ([1], p. 127):

$$S_{afb} = S_{afe} = 0 \qquad a \neq b, \quad e \neq f; \tag{3.9}a$$

(3.9)*b* *the remaining components of* S_{ijk} *remain the same as* (1.31). *The non-holonomic components of* $F_{\omega\mu\nu}$ *in this case are*

(3.10)*a* *the four components* F_{faa} *are arbitrary*

(3.10)*b* *the remaining* $F_{ijk} = 0 .$

Proof. Multiplying (3.5)*b* by $h^{\mu\nu}$ and summing with regard to μ and ν, we get in consequence of (2.12) and (3.8)*b*

$$k_\omega{}^\alpha k^{\nu\beta}(\tfrac{1}{2}K_{\alpha\beta\nu} - 3K_{[\alpha\beta\nu]}) = 2F_\omega + K_\omega . \tag{3.11}$$

Writing the left hand side of (3.11) in non-holonomic form and using (1.19)*b*, (1.28) and the fact that (HLAVATÝ [1], p. 126)

$$K_{abc} = 0 , \tag{3.12}$$

we find that this term vanishes. Hence we have (3.8)*a*.

In consequence of (3.12), the equation (3.8)a can be written in non-holonomic form as

$$K_{i(ab)}K_{i(ef)} + 2F_{i(ab)} + 2F_{i(ef)} = 0 \qquad (a \neq b,\ e \neq f)$$

which by virtue of (3.6) assumes the form

(3.13)a $$K_{a(ef)} = K_{f(ab)} = 0\,.$$

But

$$K_{aef} = \nabla_a k_{fe} + \nabla_e k_{af} + \nabla_f k_{ae} = -2\nabla_{(e}k_{f)a} = K_{a(ef)}$$

and

$$K_{fab} = \nabla_f k_{ba} + \nabla_a k_{fb} + \nabla_b k_{fa} = -2\nabla_{(a}k_{b)f} = K_{f(ab)}\,.$$

Therefore (3.12)a are equivalent to

(3.13)b $$K_{aef} = K_{fab} = 0\,.$$

Using these equations in (1.31) we obtain (3.9). Also using these equations in (3.6) we obtain (3.10).

We will now consider the tensor $U_{\omega\mu\nu}$.

Theorem (3.7). *A necessary and sufficient condition for the existence of at least one solution* $U_{\omega\mu\nu}$ *of* (1.10) *when* $k=0$, $g=2$ *is*

(3.14)a $$(\overset{(10)1}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)} - \overset{(20)2}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)} - \overset{112}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)})C_{\alpha\beta\gamma} = 0 \quad (*)$$

equivalent to

(3.14)b $$(\overset{(10)2}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)} - \overset{(20)3}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)} - \overset{113}{A}{}^{\alpha\beta\gamma}_{\omega(\mu\nu)})K_{\alpha\beta\gamma} = 0\,,$$

which in non-holonomic form reduces to the condition (1.30). *Hence the conditions* (3.4) *and* (3.14) *are equivalent.*

Proof. The condition (3.14)a is obvious from the solution (1.13). Writing for $C_{\alpha\beta\gamma}$ from (1.10)b in (3.14)a we obtain (3.14)b. Writing (3.14) in non-holonomic form and using (1.19)b and (1.28), we obtain (1.30). The remaining part of the proof is obvious.

Theorem (3.8). *When the condition* (3.14) ((3.4)) *is satisfied the solution* $U_{\omega\mu\nu}$ *of* (1.10) *has non-holonomic components which are symmetric in the last*

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* cf. to (1.10)b

two indices and which are given by

(3.15)*a* *the eight components* U_{eaa}, U_{aea} *are arbitrary*

(3.15)*b* $$4U_{aeb} = \underset{b}{\lambda} K_{aeb}$$ $a \neq b, \ e \neq f$

(3.15)*c* $$2U_{efa} = \underset{a}{\lambda} K_{efa}$$

(3.125)*d* *the remaining* U_{ifg}, U_{bcd}, U_{eab}, $U_{faf} = 0$.

Proof. Substituting S_{ijk} from (1.31) into the non-holonomic form of (1.7)*a*, we get at once (3.15). They can be obtained directly from (1.21)*a*, (1.28), (1.29) and (1.30).

Theorem (3.9). *When the condition* (3.14) ((3.4)) *is satisfied* $U_{\omega\mu\nu}$ *is of the form*

(3.16)*a* $$U_{\omega\mu\nu} = 2C_{\alpha(\mu\nu)} - k_{\omega}^{\ \alpha} C_{\alpha\beta(\nu} k_{\mu)}^{\ \ \beta} + G_{\omega\mu\nu}$$

which can be written as

(3.16)*b* $$U_{\omega\mu\nu} = K_{\omega\mu\alpha} k_{\nu}^{\ \alpha} + K_{\omega\nu\alpha} k_{\mu}^{\ \alpha} + G_{\omega\mu\nu}$$

where $G_{\omega\mu\nu}$ *is a tensor symmetric in the last two indices. The non-holonomic components of* $G_{\omega\mu\nu}$ *are symmetric in the last two indices and are given by*

(3.17)*a* G_{eaa}, G_{aea} *are arbitrary*

(3.17)*b* $$G_{aeb} = -\tfrac{3}{4} \underset{b}{\lambda} K_{aeb}$$ $(a \neq b, \ e \neq f)$

(3.17)*c* $$G_{efa} = -\tfrac{1}{2} \underset{a}{\lambda} K_{efa}$$

(3.17)*d* *the remaining* G_{ifg}, G_{bcd}, G_{eab}, $G_{faf} = 0$.

Proof. Substituting the condition (3.14) in (1.13) and putting $g = 2$, we find that the solution must be of the form (3.16)*a*.

In consequence of (1.19)*b*, (1.28), (1.10)*b* and (3.12) we get

$$k_x^{\ i} C_{ij(z} k_{y)}^{\ \ j} = k_x^{\ i} k_y^{\ j} k_z^{\ k} K_{i(jk)} = \underset{x}{\lambda} \underset{y}{\lambda} \underset{z}{\lambda} K_{x(yz)} = 0 .$$

Hence (3.16)*a* assumes the form (3.16)*b*.

Writing (3.16)*b* in non-holonomic form and using (3.15), we have (3.17).

* * *

In conclusion I take this opportunity to thank Professor HLAVATÝ for his keen interest, advice and help in the preparation of this paper.

RIASSUNTO (*)

In un recente lavoro [3] ho dato la soluzione del primo sistema di equazioni della teoria unificata dei campi di Einstein servendomi del metodo delle sostituzioni ripetute, ed ho trovato le condizioni per l'esistenza e l'unicità della soluzione per le prime due classi di $k_{\lambda\mu}$ e per tutti gli indici di inerzia di $h_{\lambda\mu}$ (+). Oggetto del presente lavoro è lo studio delle soluzioni dei casi singolari della seconda classe in cui queste condizioni non siano soddisfatte. Si deve, tuttavia, notare che HLAVATY [1] ha studiato le soluzioni singolari in forma sia olonoma che non olonoma. Nel presente lavoro le soluzioni singolari saranno date in altre forme olonome usando le soluzioni da me ottenute [3] per il caso non degenerato.

(+) Le condizioni sono sempre soddisfatte nelle classi terza e quarta, purchè tali classi esistano.
(*) *Traduzione a cura della Redazione.*

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Calcul des moments quadrupolaires des noyaux atomiques, dans le schéma de Nilsson.

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Résumé. — On a calculé la valeur moyenne de l'operator $e\sum_i r_i''^2 Y_{2,0}(\vartheta_i'', \varphi_i'')$ correspondant à un système de protons qui se meuvent independemment dans le champ nucléaire deformé. La fonction d'onde du noyau est écrite dans l'approximation ou l'on considère que le mouvement de rotation du champ nucléaire n'influence pas les mouvements individuels des nucléons. Pour la partie « individuelle » de la fonction d'onde on a employé le produit de deux determinants de Slater (un pour les protons et l'autre pour les neutrons), constitués par les fonctions d'onde uninucléoniques fournis par le modèle de Nilsson. Les calcules ont été effectués tant dans le représentation qui fait abstraction du couplage entre les états à différants N, aussi bien que dans celle qui tient compte de ce couplage. La comparaison avec les valeurs expérimentales des quadrupoles nucléaires pour les noyaux impairs qui se trouvent dans la region $150 < A < 188$, montre que l'utilisation de la dernière représentation est indispensable.

Nous avons tenté de faire le calcul rigoureux des quadrupoles nucléaires, dans la variante Nilsson [1] du modèle généralisé. L'expression donnée par cet auteur pour le quadrupole intrinsèque,

$$Q_0 \approx 0.8 Z R_0^2 \delta(1 + \tfrac{2}{3}\delta) \qquad (R_0 = 1.2 A^{\frac{1}{3}} \cdot 10^{-13} \text{ cm}) \tag{1}$$

est une expression approximative, pour l'obtention de laquelle il n'est pas fait usage des fonctions d'onde de Nilsson.

(1) S. G. Nilsson: *Det. Kong. Danske Mat.-Fys. Medd.*, **29**, 16 (1953).

Il est nécessaire de préciser dès l'abord que nous n'avons pas espéré obtenir une expression qui, comparée aux données expérimentales, fournisse des valuers du paramètre de déformation δ très différentes de celles que l'on obtient à l'aide de (1). Nous avons seulement voulu voir comment le modèle de Nilsson « fonctionne » pour le calcul des quadrupoles nucléaires (aspect de la formule du quadrupole intrinsèque, manière dont celle-ci peut être comparée à l'expérience) et comment l'expression obtenue peut éventuellement être utilisée pour le calcul d'autres grandeurs et propriétés nucléaires dans le modèle de Nilsson.

Les calculs que nous avons effectués, nous ont donné la valeur moyenne de l'opérateur quadrupole conformément au modèle admis, c'est-à-dire en partant de l'hypothèse que les nucléons accomplissent des mouvements individuels dans un champ self-consistant déformé, qui, à son tour, accomplit des mouvements rotationnels lents, qui n'affectent pas les mouvements individuels (approximation adiabatique). C'est pourquoi nous avons préféré ne pas faire usage de l'opérateur quadrupole utilisé couramment dans les calculs basés sur le modèle généralisé:

$$e\hat{Q} = \sum_i e_i r_i''^2 Y_{20}(\vartheta_i'', \varphi_i'') + \frac{3}{4\pi} ZeR_0^2 \alpha_{2,0}^*,$$

qui est écrit par rapport au système de référence K'', fixe dans l'espace. Car le paramètre $\alpha_{2,0}^*$, qui décrit la déformation du « core » est exprimé en fonction des coordonnées individuelles par la relation

$$\alpha_{2,0}^* = \frac{4\pi}{3A} \sum_{i=1}^{A} \left(\frac{r_i''}{R_0}\right)^2 Y_{2,0}(\vartheta_i'', \varphi_i''),$$

qui implique un modèle hydrodynamique du « core », ce qui est en désaccord avec les hypothèses fondamentales du modèle de Nilsson et également avec l'expérience (voir [2]). Nous avons donc cru plus correct d'utiliser l'opérateur suivant, pour la construction duquel on admet seulement qu'il provient des contributions de Z protons qui se meuvent indépendamment dans le noyau:

$$e\hat{Q} = e \sum_{i=1}^{Z} r_i''^2 Y_{2,0}(\vartheta_i'', \varphi_i''). \tag{2}$$

En tenant compte du fait que l'opérateur (2) est un tenseur irréductible de deuxième rang et en faisant usage des coordonnées sans dimensions x, y, z, ce qui comporte la relation suivante entre r et r':

$$r^2 = \frac{M\omega_0(\delta)}{\hbar} r'^2; \quad r^2 = x^2 + y^2 + z^2; \quad \omega_0(\delta) \approx \mathring{\omega}_0 \left(1 + \frac{2}{9}\delta^2\right), \tag{3}$$

([2]) A. Bohr and B. Mottelson: *Det. Kong. Danske Mat.-Fys. Medd*, **27**, 16, (1953).

on obtient pour $\hat{Q}$ l'expression suivante, par rapport au système de référence K' lié au noyau:

$$\hat{Q} = \frac{\hbar}{M\omega_0(\delta)} \sum_{i=1}^{z} r_i^2 \sum_{\nu} Y_{2,\nu}(\vartheta_i', \varphi_i') \mathcal{D}_{0,\nu}^2, \tag{4}$$

$\mathcal{D}_{0,\nu}^2$ est un élément de la représentation $2.2+1$ dimensionelle du groupe des rotations.

Nous avons calculé la valeur moyenne $Q = \langle \hat{Q} \rangle$, considérant que la fonction d'onde associée au noyau est de la forme:

$$|\Omega; JKJ\rangle = \sqrt{\frac{2J+1}{16\pi^2}} \left\{ \chi_\Omega \mathcal{D}_{J,K}^J + (-1)^{J-\sum i} \chi_{-\Omega} \mathcal{D}_{J,-K}^J \right\}. \tag{5}$$

En ce qui concerne la partie de la fonction d'onde correspondante aux mouvements individuels, nous avons admis pour elle une forme directement déterminée par les hypothèses fondamentales du modèle utilisé:

$$\chi_\Omega = \frac{1}{\sqrt{Z!}} \sum_{P_p} (-1)^{P_p} P_p \Phi_\Omega(p) \frac{1}{\sqrt{N!}} \sum_{P_n} (-1)^{P_n} P_n \Phi_\Omega(n) = \tag{6}$$

$$= \frac{1}{\sqrt{Z!}} \begin{vmatrix} \chi_{\Omega_1}(p_1)\, \chi_{\Omega_1}(p_2) \cdots \chi_{\Omega_1}(p_Z) \\ \chi_{\Omega_2}(p_1)\, \chi_{\Omega_2}(p_2) \cdots \chi_{\Omega_2}(p_Z) \\ \cdot \\ \cdot \\ \cdot \\ \chi_{\Omega_Z}(p_1)\, \chi_{\Omega_Z}(p_2) \cdots \chi_{\Omega_Z}(p_Z) \end{vmatrix} \frac{1}{\sqrt{N!}} \begin{vmatrix} \chi_{\Omega_1}(n_1)\, \chi_{\Omega_1}(n_2) \cdots \chi_{\Omega_1}(n_N) \\ \chi_{\Omega_2}(n_1)\, \chi_{\Omega_2}(n_2) \cdots \chi_{\Omega_2}(n_N) \\ \cdot \\ \cdot \\ \cdot \\ \chi_{\Omega_N}(n_1)\, \chi_{\Omega_N}(n_2) \cdots \chi_{\Omega_N}(n_N) \end{vmatrix},$$

expression où $\Phi_\Omega(p)$ et $\Phi_\Omega(n)$ représentent les fonctions suivantes:

$$\Phi_\Omega(p) = \chi_{\Omega_1}(p_1)\, \chi_{\Omega_2}(p_2) \cdots \chi_{\Omega_Z}(p_Z); \qquad \Phi_\Omega(n) = \chi_{\Omega_1}(n_1)\, \chi_{\Omega_2}(n_2) \cdots \chi_{\Omega_N}(n_N).$$

Ici $p_1, p_2, \ldots, p_z$ et $n_1, n_2, \ldots, n_N$ représentent les coordonnées spatiales et de spin des Z protons et des N neutrons et $\Omega_1, \Omega_2, \ldots, \Omega_Z$ et $\Omega_1, \Omega_2, \ldots, \Omega_N$ représentent les nombres quantiques associés. P_p et P_n indiquent des opérateurs dont l'effet sur $\Phi_\Omega(p)$ et $\Phi_\Omega(n)$ est une certaine permutation des coordonnées nucléoniques par rapport aux nombres quantiques (ou bien inversement, — tenant compte du fait qu'il s'agit exclusivement d'états antisymétriques — une certaine permutation des nombres quantiques par rapport aux coordonnées nucléoniques); quant à p_p et p_n, ils représentent respectivement la parité des permutations protoniques et neutroniques (concernant ces notations, voir [3], chap. VI).

[3] U. E. Condon and G. H. Shortley: *The Theory of Atomic Spectra* (Cambridge, 1953).

Dans le schéma de Nilsson, on fait usage, pour écrire les fonctions d'onde individuelles χ_{Ω_i}, de la représentation dans laquelle les vecteurs de base, $|N l \Lambda \Sigma\rangle$, sont les vecteurs propres du hamiltonien correspondant au mouvement dans un potentiel oscillant tridimensionnel à symétrie sphérique:

$$(7) \qquad |N l \Lambda \Sigma\rangle \approx r^l \exp\left[-\frac{r^2}{2}\right] F\left(\frac{l-N}{2}, l+\frac{3}{2}, r^2\right) Y_{l,\Lambda} f_{s,\Sigma} .$$

Dans cette représentation

$$(8) \qquad \chi_{\Omega_i} = |N_i \Omega_i \alpha_i\rangle = \sum_{l_i \Lambda_i \Sigma_i} a_{l_i \Lambda_i} |N_i l_i \Lambda_i \Sigma_i\rangle ; \qquad (-1)^{j_i} \chi_{-\Omega_i} = i^{2(\pi_i - \frac{1}{2})} |N_i - \Omega_i \alpha_i\rangle ,$$

où $\Lambda_i + \Sigma_i = \Omega_i$; π_i représente la parité de l'état χ_{Ω_i}.

Le calcul de Q, tenant compte de (5), (6), (7) et (8) et des propriétés des fonctions $\mathcal{D}$, revient au calcul des intégrales suivantes:

$$\langle \chi_\Omega | \sum_{i=1}^{Z} r_i^2 Y_{2,0} | \chi_\Omega \rangle ; \qquad \langle \chi_\Omega | \sum_{i=1}^{Z} r_i^2 Y_{2,2k} | \chi_{-\Omega} \rangle$$

$$\langle \chi_{-\Omega} | \sum_{i=1}^{Z} r_i^2 Y_{2,0} | \chi_{-\Omega} \rangle ; \qquad \langle \chi_{-\Omega} | \sum_{i=1}^{Z} r_i^2 Y_{2,-2k} | \chi_\Omega \rangle$$

dont la forme générale est $\langle \chi_\Omega | \sum_{i=1}^{Z} f(i) | \chi_{\Omega'} \rangle$. Une intégrale de cette sorte, tenant compte de (6), peut être écrite de la manière suivante (voir [3], ch. VI):

$$\langle \chi_\Omega | \sum_{i=1}^{Z} f(i) | \chi_{\Omega'} \rangle = \frac{1}{Z!} \sum_{i=1}^{Z} \sum_{P_p} \sum_{P'_p} (-1)^{P_p + P'_p} \int P_p \Phi^*_\Omega(p) f(i) P'_p \Phi_{\Omega'}(p) \, d\tau \cdot$$

$$\cdot \frac{1}{N!} \sum_{P_n} \sum_{P'_n} (-1)^{P_n + P'_n} \int P_n \Phi^*_\Omega(n) P'_n \Phi_{\Omega'}(n) \, d\tau ,$$

où le signe d'intégration signifie intégration et sommation par rapport à toutes les coordonnées spatiales et de spin des protons, respectivement des neutrons. Il est convenable de considérer P comme un opérateur qui transforme les nombres quantiques de chaque nucléon, en autres nombres quantiques. Alors

$$\int P_p \Phi^*_\Omega(p) f(i) P'_p \Phi_{\Omega'}(p) \, d\tau = \langle P_p(N_1 \Omega_1 \alpha_1) | P'_p(N'_1 \Omega'_1 \alpha'_1) \rangle \cdot$$

$$\cdot \langle P_p(N_2 \Omega_2 \alpha_2) | P'_p(N'_2 \Omega'_2 \alpha'_2) \rangle \ldots \langle P_p(N_i \Omega_i \alpha_i) | f(i) | P'_p(N'_i \Omega'_i \alpha'_i) \rangle \ldots$$

$$\ldots \langle P_z(N_Z \Omega_Z \alpha_Z) | P'_p(N'_Z \Omega'_Z \alpha'_Z) \rangle .$$

Si l'on tient compte de l'ortogonalité des fonctions $|N_i\Omega_i\alpha_i\rangle$:

$$\langle N_i\Omega_i\alpha_i|N_i'\Omega_i'\alpha_i'\rangle = \delta_{N_iN_i'}\delta_{\Omega_i\Omega_i'}\delta_{\alpha_i\alpha_i'},$$

il s'ensuit que

$$\langle\chi_\Omega|\sum_{i=1}^{Z} f(i)|\chi_{-\Omega}\rangle = \langle\chi_{-\Omega}|\sum_{i=1}^{Z} f(i)|\chi_\Omega\rangle = 0,$$

car, en ce cas $(N_i'\Omega_i'\alpha_i') = (N_i', -\Omega_i, \alpha_i')$ ou $(N_i, -\Omega_i, \alpha_i)$, donc

$$P_p'(N_i'\Omega_i'\alpha_i') \neq P_p(N_i\Omega_i\alpha_i)$$

pour n'importe quel i.

La même propriété permet d'écrire:

$$\langle\chi_\Omega|\sum_{i=1}^{Z} f(i)|\chi_\Omega\rangle = \sum_{i=1}^{Z}\langle\chi_{\Omega_i}|f(i)|\chi_{\Omega_i}\rangle.$$

Donc

$$\langle\chi_\Omega|\sum_{i=1}^{Z} r_i^2 Y_{2,0}|\chi_\Omega\rangle = \sum_{i=1}^{Z}\langle\chi_{\Omega_i}|r_i^2 Y_{2,0}|\chi_{\Omega_i}\rangle\cdot$$

$$\cdot\, i^{2J-2\sum_i(\pi_i-\frac{1}{2})}[i^{2J+2\sum_i(\pi_i-\frac{1}{2})}]^*\langle\chi_\Omega|\sum_{i=1}^{Z} r_i^2 Y_{2\,0}|\chi_{-\Omega}\rangle = \sum_{i=1}^{Z}\langle\chi_{-\Omega_i}|r_i^2 Y_{2,0}|\chi_{-\Omega_i}\rangle.$$

Des calculs simples permettent de démontrer que

$$\langle\chi_{\Omega_i}|r_i^2 Y_{2\,0}|\chi_{\Omega_i}\rangle = \langle\chi_{-\Omega_i}|r_i^2 Y_{2\,0}|\chi_{-\Omega_i}\rangle =$$

$$= \sqrt{\frac{5}{4\pi}}\sum_{l_i,l_i'}\langle l_i 200|l_i'0\rangle\sqrt{\frac{2l_i+1}{2l_i'+1}}\langle N_i l_i'|r_i^2|N_i l_i\rangle\sum_{\Lambda_i,\Sigma_i,\Sigma_i'} a_{l_i'\Lambda_i'}a_{l_i\Lambda_i}\langle l_i 2\Lambda_i 0|l_i'\Lambda_i\rangle\,\delta^{\Sigma_i'\Sigma_i}.$$

En introduisant les notations

$$\mathscr{A}_i(\delta) = \sum_{l_i,l_i'} b_{l_i,l_i'}\sum_{\Lambda_i,\Lambda_i',\Sigma_i\Sigma_i'} a_{l_i'\Lambda_i}'a_{l_i\Lambda_i}\langle l_i 2\Lambda_i 0|l_i'\Lambda_i'\rangle\delta_{\Sigma_i'\Sigma_i},$$

$$b_{l_i,l_i'} = \langle l_i 200|l_i'0\rangle\sqrt{\frac{2l_i+1}{2l_i'+1}}\langle N_i l_i'|r_i^2|N_i l_i\rangle,$$

l'on obtient l'expression suivante pour le moment quadrupolaire:

$$Q = \frac{3K^2 - J(J+1)}{(2J+3)(J+1)}Q_0,$$

où le moment quadrupolaire intrinsèque Q_0 est de la forme

$$Q_0 = \sqrt{\frac{5}{4\pi}} \frac{\hbar}{M\omega_0(\delta)} \sum_{i=1}^{z} \mathscr{A}_i(\delta) . \tag{9}$$

Les valeurs des grandeurs $\mathscr{A}_i(\delta)$ ont été calculées pour presque tous les niveaux donnés dans le Tableau I du mèmoire de Nilsson. On a fait aussi le calcul des valeurs $\mathscr{A}_i$ correspondantes aux protons situés sur les niveaux à $N = 4$. A cette fin on a fait usage des fonctions d'ondes corrigées.

Nous avons calculé également les quadrupoles nucléaires dans la représentation proposée par Nilsson ([1], appendice A), dans laquelle il est tenu compte des couplages entre les états avec N différents, qui apparaissent par la diagonalization de l'hamiltonien, couplages qui avaient été négligés dans les calculs mentionnés plus haut. Ces couplages peuvent être pris en considération en modifiant les paramètres δ et ω_0 par une transformation convenable des coordonnées. L'on introduit ainsi à la place de δ le paramètre ε, liè au précédent par la relation

$$\varepsilon = \delta + \tfrac{1}{6}\delta^2 + O(\delta^3)$$

et l'on admet que les fréquences de l'oscillateur anisotrope tridimensionnel dépendent de ε conformément aux relations suivantes:

$$\omega_x = \omega_y = \omega_0(\varepsilon)(1 + \tfrac{1}{3}\varepsilon) ; \qquad \omega_z = \varepsilon_0(\varepsilon)(1 - \tfrac{2}{3}\varepsilon) ; \qquad \omega_0(\varepsilon) = \hat{\omega}_0[1 + \tfrac{1}{9}\varepsilon^2 + O(\varepsilon^3)] .$$

L'on substitue aux coordonnées x', y', z' les coordonnées sans dimensions ξ, η, ζ, de manière que, au lieu de (3), nous avons:

$$\varrho^2 = \frac{M\omega_0(\varepsilon)}{\hbar}\left[r'^2\left(1 + \frac{1}{3}\varepsilon\right) - \varepsilon z'^2\right] \qquad \varrho^2 = \xi^2 + \eta^2 + \zeta^2 . \tag{10}$$

La diagonalisation de l'hamiltonien H est effectuée par analogie avec le procédé antérieur, en faisant usage d'une représentation dans la quelle les vecteurs de base $|N_t l_t \Lambda_t \Sigma\rangle$ sont les vecteurs propres de l'hamiltonien H_0 correspondant au mouvement dans un potentiel oscillant tridimensionnel à symétrie sphérique.

Dans cette représentation l_t et Λ_t sont les nombres quantiques correspondants aux opérateurs l_t et $(l_t)_z$, définis de la manière suivante:

$$\boldsymbol{l}_t^2 = (l_t)_x^2 + (l_t)_y^2 + (l_t)_z^2 ; \quad (l_t)_x = -i\left(\eta\frac{\partial}{\partial\zeta} - \zeta\frac{\partial}{\partial\eta}\right) \text{ etc.} .$$

N_t signifie, comme précédemment, le nombre total de quanta d'oscillation.

En utilisant cette représentation, on peut montrer que le terme H de l'hamiltonien qui représente l'interaction du nucléon avec l'axe de symétrie du noyau, a des éléments matriciels nuls entre états à nombres quantiques N_t différents et que l'opérateur $\hat{H}_0 + H_\varepsilon + C\boldsymbol{l}_t\boldsymbol{s} + D\boldsymbol{l}_t^2$ a les mêmes éléments matriciels que l'opérateur $\hat{H}_0 + H_\delta + C\boldsymbol{l}\boldsymbol{s} + D\boldsymbol{l}^2$ dans la représentation $|Nl\Lambda\Sigma\rangle$, avec la seule différence que les paramètres δ et $\omega_0(\delta)$ deviennent ε et $\omega_0(\varepsilon)$ dans la nouvelle représentation. On démontre aussi que les effets du terme perturbateur de l'hamiltonien $H_{\text{pert}} = C(\boldsymbol{l} - \boldsymbol{l}_t)\boldsymbol{s} + D(\boldsymbol{l}^2 - \boldsymbol{l}_t^2)$ sont petits et que leurs contributions aux valeurs moyennes et aux éléments matriciels ne sont pas importantes.

Dans la nouvelle représentation, les coefficients du Tableau I de [1] sont interprétés comme étant les coefficients des fonctions d'onde dans la représentation $|N_t l_t \Lambda_t \Sigma\rangle$.

En effectuant la transformation (10) et en tenant compte du fait que ϱ_i^2 est un tenseur irréductible de rang zéro, on peut écrire pour Q, dans le système de référence lié au noyau,

$$\hat{Q} = \frac{\hbar}{M\omega'(\varepsilon)} \sum_{i=1}^{Z} \left[\varrho_i^2 \sum_\nu Y_{2,\nu} \mathcal{D}^2_{0,\nu} + \frac{\sqrt{5}}{3} \varepsilon \varrho_i^2 \sum_\nu Y_{0,\nu} \mathcal{D}^2_{0,\nu} \right].$$

L'on obtient pour Q_0 l'expression suivante:

$$Q_0 = \frac{\hbar}{M\omega(\varepsilon)} \sqrt{\frac{5}{4\pi}} \sum_{i=1}^{Z} \left[\mathcal{A}_i(\varepsilon) + \frac{2}{3} \varepsilon \langle N_i l_i | \varrho_i^2 | N_i l_i \rangle \right], \tag{11}$$

où $\omega_0'(\varepsilon) = \omega_0(1 - \frac{1}{3}\varepsilon - \frac{1}{9}\varepsilon^2)$. Les grandeurs $\mathcal{A}_i$ ont les mêmes expressions que celles qui intervenaient dans la représentation précédente; les $\langle N_i l_i | \varrho_i^2 | N_i l_i \rangle$ sont donnés par (11a) de [1].

Comparaison avec l'expérience.

Les expressions (9) et (11) ont été comparées aux valeurs de Q_0 obtenues par des expériences d'excitation coulombienne, pour les noyaux impairs de la région $150 < A < 188$, valeurs fournies per [4]. A cet effet nous nous sommes servi des résultats obtenus par NILSSON dans le calcul de l'énergie totale du noyau et de sa déformation d'équilibre. En admettant qu'entre les nucléons s'exercent exclusivement les interactions typiques du problème de deux corps, on obtient l'expression suivante pour l'énergie totale $\mathcal{H}(\varepsilon)$ du noyau:

$$\mathcal{H}(\varepsilon) = \tfrac{3}{4}\hbar\omega_0 \left[\sum_{i=1}^{A} (N_i + \tfrac{3}{2})(1 + \tfrac{1}{9}\varepsilon^2) + \sum_{i=1}^{A} \varkappa r_i(\varepsilon) + \tfrac{1}{4} \sum_i \langle U_i \rangle \right],$$

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[4] A. BOHR and B. MOTTELSON: *Det. Kong. Danske Mat.-Fys. Medd.*, **30**, 1 (1955).

où $U \approx Cl_i s + Dl_i^2$ et avec $\varkappa = 0.05$ pour les niveaux neutroniques et $\varkappa = 0.0613$ pour les niveaux protoniques. La déformation d'équilibre ε_e du noyau peut être obtenue en résolvant l'équation

$$\frac{\partial \mathcal{H}(\varepsilon)}{\partial \varepsilon} = 0 .$$

Les valeurs de la déformation d'équilibre, obtenues de la sorte, sont représentées graphiquement en ([5]). Nous avons considéré uniquement les noyaux pour lesquels la déformation d'équilibre ne dépasse pas sensiblement l'intervalle $4 \leqslant \eta \leqslant 6$, car le schéma de Nilsson ne peut être considéré comme une bonne approximation de la structure réelle des noyaux, que pour les grandes déformations.

Les résultats obtenus sont donnés dans le tableau suivant:

Noyau	$^{153}_{63}$Eu	$^{159}_{65}$Tb	$^{165}_{67}$Ho	$^{175}_{71}$Lu	$^{179}_{72}$Hf	$^{177}_{72}$Hf	$^{169}_{69}$Tm
$Q_{0\,\mathrm{exp}}$ (en 10^{-24} cm^2)	7.7	8.4	8.6	8.8	8.5	8	8.1
δ_e	0.33	0.34	0.30	0.26	0.24	0.25	0.3
$\hbar\mathring{\omega}_0(Q_0(\delta_e))$ (MeV)	4.52	4.68	4.21	3.71	3.52	3.83	4.52
ε_e	0.348	0.359	0.315	0.348	0.249	0.26	0.315
$\hbar\mathring{\omega}_0(Q_0(\varepsilon_e))$ (MeV)	12.10	13.29	11.73	10.23	9.73	10.17	12.72
$\hbar\mathring{\omega}_0 = 41\,A^{-\frac{1}{3}}$ (MeV)	7.67	7.57	7.48	7.33	7.28	7.30	7.42

Nous avons noté $\hbar\mathring{\omega}_0(Q_0(\delta_e))$ et $\hbar\mathring{\omega}_0(Q_0(\varepsilon_e))$ les valeurs de $\hbar\mathring{\omega}_0$ obtenues des expressions de Q_0, calculées dans la représentation approximative de Nilsson, respectivement dans la représentation corrigée. Il resulte du tableau que l'apparition du second terme dans l'expression (11), ainsi que la modification de la constante ω_0 change sensiblement les valeurs obtenues pour $\hbar\mathring{\omega}_0$. Cela signifie que l'utilisation de la représentation corrigée est indispensable pour le calcul des moments quadrupolaires, ce que d'ailleurs NILSSON avait lui-même supposé ([1]), appendice A). Nous avons indiqué aussi, à titre de comparaison, les valeurs obtenues à partir de l'expression $\hbar\mathring{\omega}_0 = 41A^{-\frac{1}{3}}$ MeV déduite dans l'hypothèse que

(12) $$\langle N_i l_i | r'^2_i | N_i l_i \rangle \approx \tfrac{3}{5} R_0^2$$

([5]) B. MOTTELSON and S. G. NILSSON: *Phys. Rev.*, **99**, 1615 (1955).

pour tous les i compris dans l'intervalle $1 \leqslant i \leqslant Z$. Nous avons toutefois renoncé à cette hypothèse parceque les intégrales $\langle N_i l_i | r_i'^2 | N_i l_i \rangle$ peuvent être calculées dans le modèle de Nilsson pour chaque proton i. Plus encore, la formule $\hbar\omega_0 = 41 A^{-\frac{1}{3}}$ MeV conduit, pour les noyaux légers à des valeurs de $\hbar\omega_0$ en contradiction avec les valeurs obtenues à l'aide de la densité ϱ calculée au moyen des fonctions d'onde de l'oscillateur harmonique isotrope et à l'aide des valeurs expérimentales du rayon nucléaire. En effet, par cette dernière méthode, on obtient pour les noyaux ${}_2^4\mathrm{He}$, ${}_8^{16}\mathrm{O}$ et ${}_{20}^{40}\mathrm{Ca}$ une valeur de $\hbar\omega_0$ d'approximativement 5 MeV (voir (6)) tandis que la formule $\hbar\omega_0 = 41 A^{-\frac{1}{3}}$ MeV conduit pour les mêmes noyaux à une valeur moyenne de 18 MeV.

Cette contradiction est un argument contre l'approximation (12) car le modèle des couches dans la variante de l'oscillateur harmonique isotrope conduit à des fonctions propres et des valeurs propres de l'énergie proches de la réalité pour les noyaux légers.

Les valeurs ainsi obtenues peuvent être utilisées pour le calcul d'autres grandeurs et propriétés nucléaires dans lesquelles intervient la grandeur $\hbar_0\omega$. De telles grandeurs sont, par exemple, dans le schéma de Nilsson, les probabilités d'émission β et γ. Dans un travail ultérieur nous essayerons d'utiliser les résultats obtenus ici, pour le calcul d'autres grandeurs.

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(6) W. HEISENBERG: *Theorie des Atomkerns* (Göttingen 1951).

RIASSUNTO (*)

Si è calcolato il valore medio dell'operatore $e \sum_i r_i''^2 Y_{2,0}(\vartheta_i'', \varphi_i'')$ corrispondente a un sistema di protoni in moto indipendente nel campo nucleare deformato. La funzione d'onda del nucleo si scrive nell'approssimazione che risulta dalla presunzione che il moto di rotazione del campo nucleare non influisca sui moti individuali dei nucleoni. Per la parte « individuale » della funzione d'onda si è impiegato il prodotto di due determinanti di Slater (uno per i protoni e uno pei neutroni), formati con le funzioni d'onda mononucleoniche fornite dal modello di Nilsson. I calcoli sono stati eseguiti sia astraendo dall'accoppiamento tra gli stati con N differenti, sia tenendone conto. Il confronto coi valori sperimentali dei quadrupoli nucleari per i nuclei dispari che si trovano nella regione $150 < A < 188$ mostra che il ricorso alla seconda di tali rappresentazioni è indispensabile.

(*) *Traduzione a cura della Redazione.*

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A Model for Multiple Meson Production in Nucleon-Nucleon Collisions.

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Summary. — The double lump structure of the produced mesons and the persistency of the original nucleons in high energy nuclear interaction are found. An empirical model of multiple meson production in a nucleon-nucleon collision is constructed on these experimental bases. Assuming the symmetry about the two colliding nucleons in C.M.S. the collision is divided into two parts belonging to each nucleon. And, a physical quantity named « momentum of interaction » is introduced, which is the momentum difference between the initial and final state in each part. This quantity is considered to be a parameter representing the magnitude of the effect of the interaction in the collision. The average value of this quantity in high energy jets are found to be of the order of unity in nucleon mass units independently from the incident energy. Taking as a working hypothesis an empirical relation between the « momentum of interaction » and the recoil momentum of the target nucleon in L.S. which is suggested in the analysis of one high energy jet, the details of the collision are determined in our model by the magnitude of the incident energy and that of the « momentum of interaction ». This model can explain the variety of the experimental data of high energy jets which have been accumulated up to now. The physical significance of the basic assumptions of this model should be searched for in future.

1. – Introduction.

Much experimental information on the phenomena of multiple meson production in high energy nuclear interaction has been accumulated at present by the observation of jet showers produced by extremely high energy cosmic rays in photographic emulsions.

The multiplicity and angular distribution of produced mesons and the inelasticity of these jets vary widely jet by jet even when we restrict ourselves to a definite region of energy and to those events which probably correspond to nucleon-nucleon collisions. This situation suggests that the above parameters characterizing the interaction could not be simple functions of the incident energy of a nucleon only, but they should be complicated functions of at least two independent variables including the incident energy.

On the other hand, the average values of these parameters have been obtained from the results of investigation on extensive air showers produced by extremely high energy cosmic rays. And judging from the average value of the inelasticity which is much smaller than unity, it seems possible that the primary nucleon maintains its individuality and a good fraction of its energy during successive nuclear interactions.

None of various ready-made phenomenological models [1] of multiple meson production put forward so far seems to be able to explain all the observed features by itself. In order to get a complete model that can explain the variety of phenomena and has close connection with the true nature of the elementary interaction, it is, first of all, necessary to seek for another variable independent from the incident energy of a nucleon. A preliminary attempt along this line will be made in this paper, starting from the reanalysis of the experimental data.

As a starting point of our work, attention was focussed on the transverse momentum of the produced mesons with respect to the shower axis. A very interesting feature of the transverse momentum was pointed out by NISHIMURA [2] in 1955. That is; its average value is constant and independent from the primary energy, and is of the order of several hundred MeV/c. In order to study the significance of this situation, the relation between the transverse momentum and the longitudinal momentum of the produced mesons was first investigated. In the beginning of 1956, there were only two high energy jets in which the energy of the secondary particles were well determined and which were satisfactory for our analysis. These jets were observed by GLASSER *et al.* [3] and HOPPER *et al.* [4]. Analysing these two jets we found that the average value of the longitudinal component of the relative momentum of the mesons to the center of gravity of each group which was emitted in forward and backward directions in C.M.S. was of the order of that of the transverse

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[1] E. FERMI: *Prog. Theor. Phys.*, **5**, 570 (1950); W. HEISENBERG: *Kosmische Strahlung* (Berlin, 1953) 148; H. W. LEWIS, T. R. OPPENHEIMER and S. A. WOUTHUYSEN: *Phys. Rev.*, **73**, 127 (1948); S. TAKAGI: *Prog. Theor. Phys.*, **1**, 123 (1952); W. L. KRAUSHAAR and L. J. MARKS: *Phys. Rev.*, **93**, 326 (1954); L. D. LANDAU: *Dokl. Ak. Nauk* (*U.S.S.R.*), **17**, 51 (1953).

[2] J. NISHIMURA: *Soryūsiron Kenkyū*, **12**, 24 (1956), in Japanese.

[3] V. D. HOPPER, S. BISWAS and J. F. DARBY: *Phys. Rev.*, **84**, 457 (1951).

[4] R. G. GLASSER, D. M. HASKIN and M. SCHEIN: *Phys. Rev.*, **99**, 1555 (1955).

momentum. Thus the usefulness of the image that two meson lumps are generated by the collision of two nucleons and from the two lumps the secondary mesons fly out at random with about the same momentum was recognized.

The persistency of the original nucleons during the collision was also found in the analysis of the two jets. Constructing an empirical model of multiple meson production in high energy nucleon-nucleon collisions on these experimental bases, we introduced a parameter named « momentum of interaction » which was the momentum difference between initial and final state in each nucleon side in C.M.S. We regarded this quantity as a parameter which represents the magnitude of the effect of the interaction and got a model in which the final details of the collision could be derived when the incident energy and the magnitude of the « momentum of interaction » were given. This model was reported by the author in 1957 ([5]).

In the beginning of 1958, many experimental data of high energy jets were obtained by the Japanese E.C.C. group ([6]) and the Bristol group ([7]). From these data the average transverse momentum of π-mesons turns out to be ~ 0.4 GeV/c irrespective of the primary energy, and Nishimura's theory was confirmed. Therefore, we examined the applicability of our model to these high energy jets

In this paper the results of this examination are given, together with the details of our model. In Sect. **2** the process of the derivation of the four-block model (two meson lumps and two original nucleons) from the above two jets is described and the applicability of the model to other jets is examined. In Sect. **3** the « momentum of interaction » is introduced and its average value together with its relation to other parameters derived from the data of high energy jets are given. In Sect. **4** an empirical relation between the « momentum of interaction » and the recoil momentum of the target nucleon in L.S. is taken as a working hypothesis in order to describe the final details of the collision by this model. The preliminary results of the comparison between the predictions from this model and the experimental data on jets are also presented.

Recently, Ciok *et al.* ([8]) and Cocconi ([9]) also reached, independently from us, models which are similar to the first part of ours.

([5]) K. Niu: Lecture at the annual meeting of the Physical Society of Japan in Oct. 1957; K. Niu: *Uchusen Kenkyū*, **3**, 85 (1958) in Japanese; K. Niu: Lecture at the semiannual meeting of the Physical Society of Japan in May 1958.

([6]) O. Minakawa, Y. Nishimura, H. Yamanouchi, H. Aizu, H. Hasegawa, Y. Ishii, S. Tokunaga, Y. Fujimoto, S. Hasegawa, J. Nishimura, K. Niu, K. Nishikawa, K. Imaeda and M. Kazuno: *Observation of high energy jets with emulsion chambers. - Part I. Transverse momentum of* π^0*-meson* preprint).

([7]) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

([8]) P. Ciok. T. Coghen, J. Gierula, R. Holyoroki, A. Jurah, M. Mięsowicz, J. Saniewska and O. Stanisz: *Nuovo Cimento*, **8**, 166 (1958).

([9]) G. Cocconi: *An empirical model for ultrarelativistic nucleon-nucleon collisions* (preprint).

2. – Double lump structure of the produced mesons and persistency of the original nucleons in high energy nuclear interactions.

2·1. *The derivation of the four-block model from two high energy nucleon-nucleon collisions.* – In order to study the significance of the fact that the transverse momentum of the mesons is constant, the relation between the longitudinal and the transverse momenta of the produced mesons in two jets which had been observed by GLASER *et al.* and HOPPER *et al.* was investigated. Examining the relation between energies and emitting angles of the produced mesons of the two jets, we noticed the qualitative feature that, in C.M.S., the energy of the mesons emitted in a small angle (near to 0° and 180°) can take larger values than the energy of those emitted in a larger angle. This situation is shown in Fig. 1. And we found that it was convenient to assume, in order to explain the above feature, that those mesons emitted in forward and backward direction belong respectively to separated groups. Actually, we got the following relation from the results of calculation of the relative momentum of mesons to the center of each group, P^{**}, which are shown in Fig. 2,

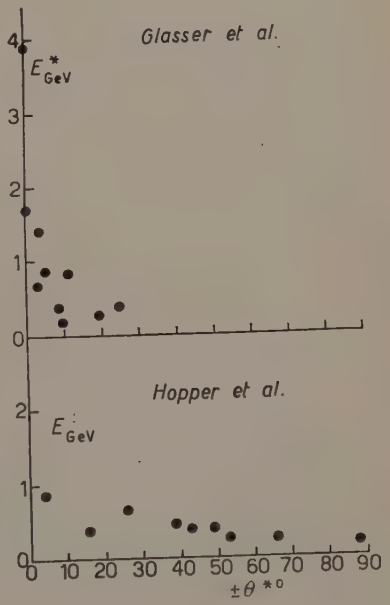


Fig. 1. – The energy versus emission angle plots for the mesons of two jets that could be examples of nucleon-nucleon collisions. GLASSER *et al.*: *Phys. Rev.*, **99**, 1555 (1955). HOPPER *et al.*: *Phys. Rev.* **84**, 457 (1951). E^*: Energy of mesons in C.M.S. θ^*. Emission angle of mesons in C.M.S.

$$\langle P_{\parallel}^{**} \rangle \approx \langle P_{\perp} \rangle \,, \tag{2.1}$$

where $P_{\parallel}^{**}$ is the longitudinal component of P^{**}. If this relation is always held in high energy jets, the following image is useful for the investigation of the problem of multiple meson production in nucleon-nucleon collisions.

In the C.M.S. of a nucleon-nucleon collision, two meson lumps are generated and each starts to travel in the opposite direction to the other. From these meson lumps individual mesons fly out at random with relative momentum to the center the order of which is equal to the average value of P_{-}. As for the explanation of the experimental fact that the transverse momentum is con-

stant and independent from the primary energy, it is necessary to assume that these meson lumps travel along the direction of the incoming nucleons and that the relative momentum of the mesons to the center of a lump is constant and independent from the other quantities. Hereafter these meson lumps will be characterized by their mass, M_M, and their Lorentz factor, γ_M. The mass of the lump is defined as

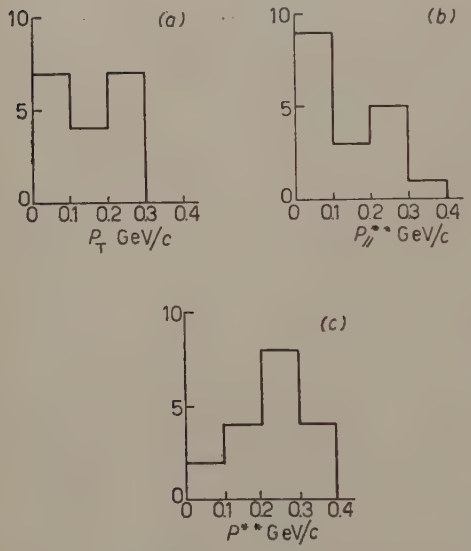


Fig. 2. – The distribution of *a*) the transverse momentum of mesons, *b*) the longitudinal component of the relative momentum of mesons with respect to the center of gravity of the group which is emitted forward or backward in C.M.S. and *c*) the relative momentum of mesons in each group, of the two jets adopted in Fig. 1.

$$M_M = \tfrac{3}{2} n_s \cdot \tfrac{1}{2} \cdot E^{**} \approx \tfrac{3}{4} n_s \cdot P \,, \tag{2.2}$$

where the suffix M indicates the meson lump.

Introducing a meson lump, following three co-ordinate systems appear in our analysis of the nucleon-nucleon collision.

L.S. : Rest system of one of the nucleons.

C.M.S.: Rest system of the center of gravity of two colliding nucleons.

M.S. : Rest system of the center of gravity of each meson lump.

To make distinction among the three systems: the suffix ** will be put on the physical quantity when it is treated in M.S.: the suffix * when in C.M.S.: and nothing when in L.S.

Next, our analysis proceeded to the search for the nucleons after the collision. Using M_M, γ_M^* and γ_N^*, the inelasticity, K^*, was expressed as follows, in nucleon mass units.

$$\gamma_N^* = \gamma_0^* - \gamma_M^* \cdot M_M \,, \tag{2.3}$$

$$K^* = \frac{\gamma_M^* \cdot M_M}{\gamma_0^* - 1} = \frac{\gamma_0^* - \gamma_N^*}{\gamma_0^* - 1} \,. \tag{2.4}$$

γ_N^*; Lorentz factor of the nucleon after the collision in the C.M.S.

The values γ_N^*, γ_M^*, M_M and K^* of the two jets are tabulated in Table I.

TABLE I.

Author	γ_0^*	γ_N^*	γ_M^*	M_M	K^*
HOPPER *et al.*	22	20	1.5	1.4	0.1
GLASSER *et al.*	100	85	5	3	0.15

Table I suggests that

$$\gamma_N^* \gg \gamma_M^* . \tag{2.5}$$

This was confirmed by an alternative way as follows. The G jet had one grey proton in the L.S. Assuming that the grey proton was a target of the collision, it had the energy of about 80 GeV in the C.M.S. This value coincided with that of Table III estimated from γ_M^* and M_M. This situation compelled to exclude the image that a nucleon was the center of the lump. Therefore, the assumption that original nucleons remain after the collision and are discriminated from meson lumps became possible.

Now we reached an empirical model of multiple meson production. The characteristics of this model were itemized as follows.

a) During the collision two meson lumps are generated.

b) Two meson lumps travel in opposite directions along the incident line of colliding nucleons.

c) From each lump individual mesons emerge at random with momentum on the average, several hundred MeV/c.

d) Two original nucleons persist throughout the collision and travel in opposite directions apart from the generated meson lumps.

2·2. *Kinematical relation between the angular distribution of mesons and the characterizing parameters of our model.* – Before proceeding with our analysis let us consider the kinematical relation between the angular distribution of mesons and the characterizing parameters of our model. If we assume isotropic emission of mesons with equal momentum in the M.S., the following asymptotic equations are satisfied on the kinematics of the decay of a meson lump.

$$\begin{cases} \operatorname{tg}\theta_{f\cdot i} = \dfrac{1}{\gamma_{Mf}} \dfrac{\sin\theta_{f\cdot i}^{**}}{\cos\theta_{f\cdot i}^{**} + \beta_{Mf}/\beta_{f\cdot i}^{**}} \approx \dfrac{1}{\gamma_{Mf}} \operatorname{tg}\dfrac{\theta_{f\cdot i}^{**}}{2}, \\ \gamma_{Mf} \operatorname{tg}\theta_{f\cdot\frac{1}{2}} \approx 1, \\ \dfrac{\operatorname{tg}\theta_{f\cdot i}}{\operatorname{tg}\theta_{f\cdot\frac{1}{2}}} \approx \operatorname{tg}\dfrac{\beta_{f\cdot i}^{**}}{2} = \operatorname{tg}\left(\tfrac{1}{2}\operatorname{arctg}\dfrac{P_{T\cdot i}}{P_{\parallel i}^{**}}\right), \end{cases} \tag{2.6}$$

where: β: velocity;
i: means the i-th particle in a meson lump.

Hereafter let us distinguish two meson lumps appearing in our model by the suffixes f and b. Then, assuming forward backward-symmetry in C.M.S., following relations are obtained,

$$
(2.7)\qquad
\begin{cases}
\gamma_{fM}\cdot\gamma_{Mb}=\dfrac{1}{\operatorname{tg}\theta_{f\frac{1}{2}}\operatorname{tg}\theta_{b\frac{1}{2}}}=\dfrac{1}{\operatorname{tg}\theta_{\frac{1}{4}}\operatorname{tg}\theta_{\frac{3}{4}}}=\gamma_0^{*2}\,,\\
\gamma^*_{Mf}=\gamma^*_{Mb}=\gamma^*_M\,,\\
\gamma^*_M\operatorname{tg}\theta^*_{\frac{1}{4}}=-\gamma^*_M\operatorname{tg}\theta^*_{\frac{3}{4}}=1\,,\\
\dfrac{\operatorname{tg}\theta_{\frac{1}{4}}}{\operatorname{tg}\theta_{\frac{1}{2}}}=\operatorname{tg}\dfrac{\theta^*_{\frac{1}{4}}}{2}\,,\\
\gamma^*_M=\dfrac{1}{\operatorname{tg}\theta^*_{\frac{1}{4}}}=\dfrac{1-\operatorname{tg}(\theta^*_{\frac{1}{4}}/2)}{2\cdot\operatorname{tg}(\theta^*_{\frac{1}{4}}/2)}=\dfrac{1}{2}\left(\dfrac{1}{\operatorname{tg}(\theta^*_{\frac{1}{4}}/2)}-\operatorname{tg}\dfrac{\theta^*_{\frac{1}{4}}}{2}\right)\\
\qquad=\dfrac{1}{2}\left(\dfrac{\operatorname{tg}\theta_{\frac{1}{2}}}{\operatorname{tg}\theta_{\frac{1}{4}}}-\dfrac{\operatorname{tg}\theta_{\frac{1}{4}}}{\operatorname{tg}\theta_{\frac{1}{2}}}\right).
\end{cases}
$$

Using the relation (2.6) information about the angular distribution of mesons in M.S. is obtained from the average values $\langle(\operatorname{tg}\theta_{f\cdot i})/(\operatorname{tg}\theta_{f\cdot\frac{1}{2}})\rangle$.

If we take an alternative model in which the transverse momentum is constant and independent from θ^{**}, we can get from it the information about $\langle P^{**}\rangle$.

2·3. *Applicability of our model to other nuclear interactions.* – As described in the preceding section, our model was derived from only two jets. In this section let us examine the applicability of our model to other nuclear interactions, using the angular distribution of mesons produced in them. Having no strict criterion to distinguish pure nucleon-nucleon collisions from others, we shall consider as nucleon-nucleon collisions those jets which are produced either by a neutral or by a single ionizing particle and which have no or few heavy prongs.

2·3.1. The existence of meson lumps. – If we observe the decay of our meson lump in L.S. and plot the angle of an individual meson on a lg tg θ scale, the values of lg tg $\theta_{i\cdot f(b)}$ will distribute, in first approximation, symmetrically with their center at lg tg $\theta_{\frac{1}{2}f(b)}$. Actually, as it is clear in the examples shown in Fig. 3, in most of the jets two centers exist in the lg tg θ distribution. The distance between these two centers, however, varies jet by jet, and in some cases two lumps nearly overlap.

When the distance of the two centers is small we cannot distinguish the mesons belonging to one meson lump from the ones belonging to the other. Then we cannot directly calculate the mean value of $(\lg\operatorname{tg}\theta_i/\operatorname{tg}\theta_{\frac{1}{2}})$ in each lump. Therefore, instead of its average value, the root mean square value

of $(\lg(\operatorname{tg}\theta_i/\operatorname{tg}\theta_{\frac{1}{2}}))$ of each jet was calculated in order to eliminate the overlapping effect of the two lumps. Let us put $\lg \operatorname{tg}\theta_{if} = x_{if}$ and $\lg \operatorname{tg}\theta_{ib} = x_{ib}$ respectively.

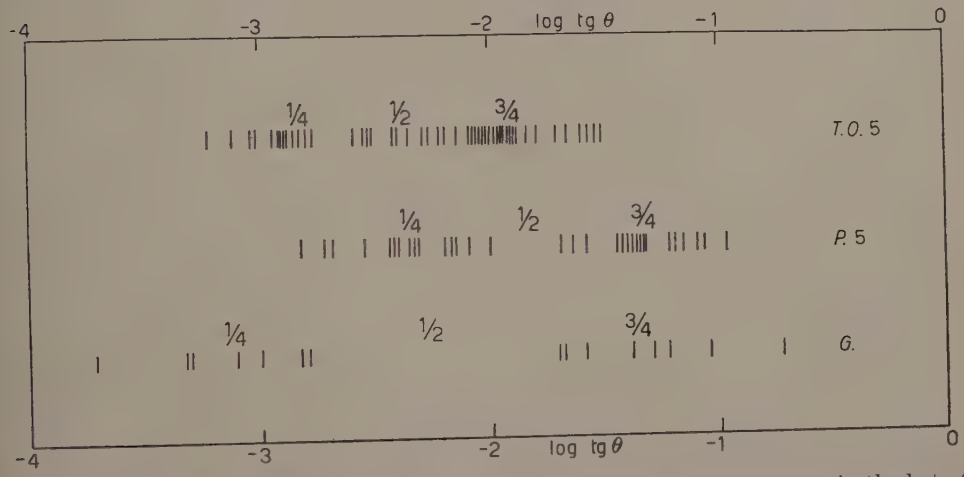


Fig. 3. – The $\lg \operatorname{tg}\theta$ plots of some high energy jets which have two centers in the $\lg \operatorname{tg}\theta$ distribution. T.O.5 Japanese E.C.C. group (private communication 1598). P.5 Bristol group (private communication 1958). G. GLASSER *et al.* adopted in Fig. 1.

Then,

$$\left\langle \left(\lg \frac{\operatorname{tg}\theta_i}{\operatorname{tg}\theta_{\frac{1}{2}}}\right)^2 \right\rangle = \langle (x_i - x_{\frac{1}{2}})^2 \rangle = \frac{1}{n_f + n_b}\left\{ \sum_1^{n_a} (x_{if} - x_{\frac{1}{2}f})^2 + \sum_1^{n_b} (x_{ib} - x_{\frac{1}{2}b})^2 \right\} =$$

$$= \frac{1}{n_f + n_b}\left\{ \sum_1^{n_f} (x_{if} - x_{\frac{1}{2}f})^2 + \sum_1^{n_b} (x_{ib} - x_{\frac{1}{2}f})^2 - n_b (x_{\frac{1}{2}b} - x_{\frac{1}{2}f})^2 - 2 \sum_1^{n_b} (x_{ib} - x_{\frac{1}{2}b})(x_{\frac{1}{2}b} - x_{\frac{1}{2}f}) \right\}.$$

The last term on the right hand side of this equation vanishes because, in the first approximation, x_{ib} distribute symmetrically with respect to $x_{\frac{1}{2}b}$. Finally we get,

$$\langle (x_i - x_{\frac{1}{2}})^2 \rangle = \frac{1}{n_a + n_b} \sum_1^{n_a + n_b} (x_i - x_{\frac{1}{2}f})^2 - \frac{n_b}{n_a + n_b} (x_{\frac{1}{2}b} - x_{\frac{1}{2}f})^2 .$$

Therefore, assuming that equal numbers of mesons are emitted from each lump, the root mean square values of $(\lg(\operatorname{tg}\theta_i/\operatorname{tg}\theta_{\frac{1}{2}}))$ of each jet can be calculated irrespective of the knowledge to which lump each meson belongs. The results of calculation are shown in Fig. 4, and $\langle (\lg(\operatorname{tg}\theta_i/\operatorname{tg}\theta_{\frac{1}{2}}))^2 \rangle^{\frac{1}{2}}$ is a constant free from $\lg(\operatorname{tg}\theta_{\frac{1}{2}\cdot b}/\operatorname{tg}\theta_{\frac{1}{2}\cdot f})$. It must be also noted that $\langle (\lg(\operatorname{tg}\theta_i/\operatorname{tg}\theta_{\frac{1}{2}}))^2 \rangle^{\frac{1}{2}}$ is independent from the incident energy of a nucleon, because those of the jets adopted in Fig. 4 range very widely from $\sim 10^{11}$ eV to $\sim 10^{14}$ eV.

From the calculated value of $\langle(\lg(\mathrm{tg}\,\theta_i/\mathrm{tg}\,\theta_{\frac{1}{2}}))^2\rangle^{\frac{1}{2}}_{f(b)}$ the value of $\langle\lg(\mathrm{tg}\,\theta_{\frac{1}{4}}/\mathrm{tg}\,\theta_{\frac{1}{2}})\rangle_{f(b)}$ and further $\langle\theta^{**}_{\frac{1}{4}}\rangle_{f(b)}$ are estimated assuming gaussian distribution of the value of $(\lg\mathrm{tg}\,\theta_i)_{f,b)}$. And the result of the estimation is

$$\langle\theta^{**}_{\frac{1}{4}}\rangle_{f(b)} \approx 53^\circ .$$

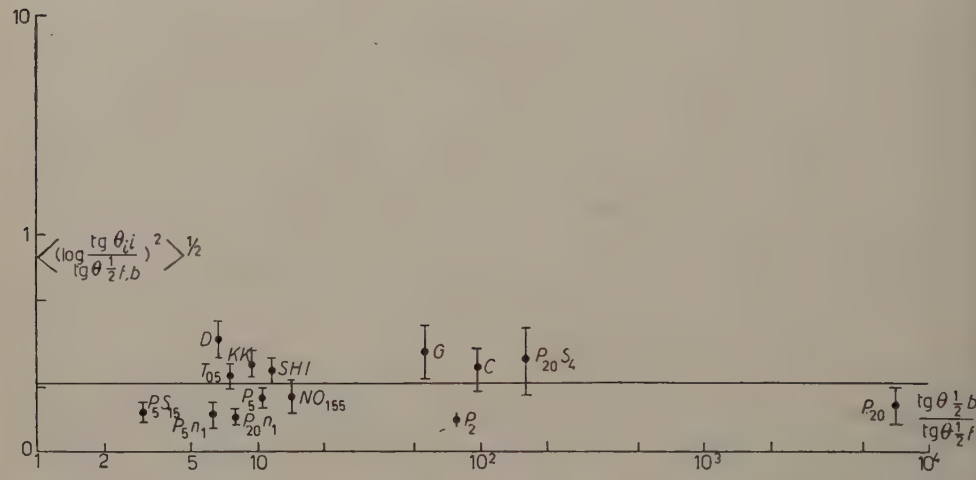


Fig. 4. – The $\langle(\lg\mathrm{tg}\,\theta_i/\mathrm{tg}\,\theta_{\frac{1}{2}f(b)})^2\rangle^{\frac{1}{2}}$ versus $(\mathrm{tg}\,\theta_{\frac{1}{2}b}/\mathrm{tg}\,\theta_{\frac{1}{2}f})$ plots for a high energy jet whose angular distribution of secondaries is well known. P_5, P_{5n1}, P_{5S15}, P_{20}, P_{20n1}, P_{20S4}, P_2; Bristol group (private communication 1958). *D*, Debenedetti *et al.*: *Nuovo Cimento*, **4**, 1142 (1956). *C*, No. 155; Ciok *et al.*, (private communication 1958). *G*, Glasser *et al.*: *Phys. Rev.*, **99**, 1555 (1955). *KK*, Kaplon and Koshiba: *Phys. Rev.*, **97**, 193 (1955). *To. 5*, *SH*1; Japanese E.C.C. group (private communication 1958).

This result shows that the angular distribution of mesons in M.S. is always nearly isotropic. If we take an alternative model in which the transverse momentum is independent from the angle in M.S., the average longitudinal momentum in M.S. would be of the order of the transverse momentum itself. This fact is a strong evidence of the existence of meson lumps.

3·3.2. The persistency of original nucleons. – Next the values of γ^*_M and γ^*_N of each jet were calculated using the relation (2.8), (2.1) and (2.3). The result of calculation is shown in Fig. 5, where γ^*_M is plotted versus γ^*_N. As is seen from this figure, γ^*_M is always much smaller than γ^*_N.

In the above calculation M_M is estimated assuming that all mesons are mesons. Taking into account K-mesons, however, the above situation would not change unless the number of K-mesons exceed $\sim 20\%$ of that of all mesons and the transverse momentum of K-meson exceeds ~ 1.5 GeV/c as reported by the Bristol group.

From the above situation it is natural to think that, in general, original nucleons persist throughout the collision and fly out apart from the meson lump.

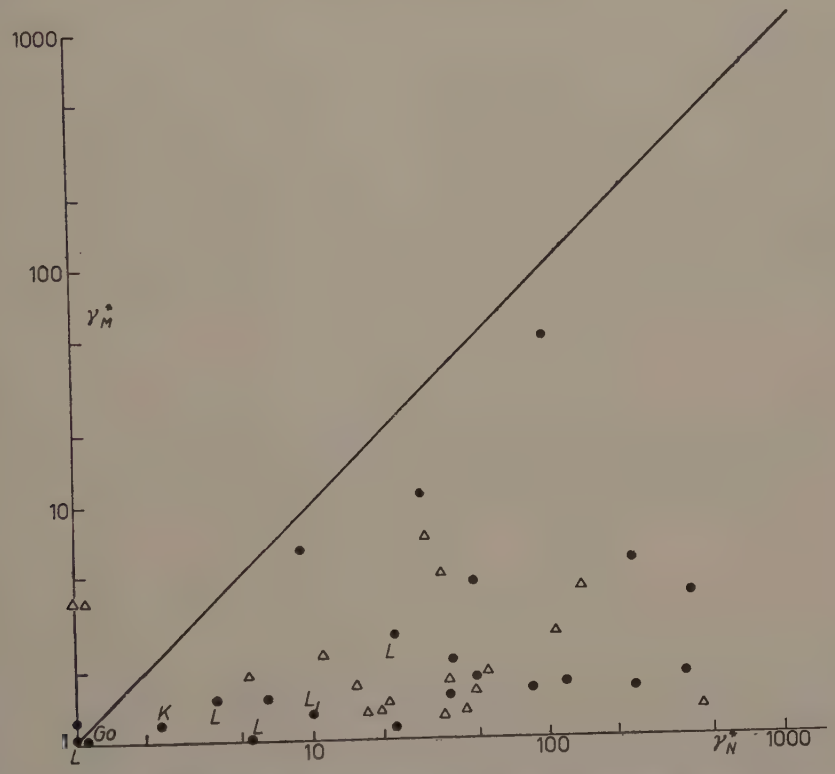


Fig. 5. – The γ_M^* versus γ_N^* plots for high energy jets. *K*. KANEKO *et al.*: *Journ. Phys. Soc. Japan*, **11**, 1129 (1956). *L*, LOHRMANN: *Nuovo Cimento*, **5**, 1074 (1957). *Go*, GOTTSTEIN and TEUCHER: *Zeit. f. Naturforsch.*, **8***a*, 120 (1953). *Li*, LINDERN: *Zeit. f. Naturforsch.*, **11***a*, 240 (1956). Δ, KAPLON and RITSON: *Phys. Rev.*, **85**, 932 (1952). The other points correspond to those jets which are adopted in Figs. 1, 2, 3 and 4.

Our empirical model of multiple meson production has been shown to be compatible with general nucleon-nucleon collisions by the results obtained in this section.

3. – The « momentum of interaction ».

3·1. *The definition of the « momentum of interaction »*. – In the preceding section one empirical model of multiple meson production has been introduced. Next we should like to investigate the correlation among the various parameters characterizing the collision.

Actual phenomena are, however, very complicated and even if we restrict ourselves to a definite region of the energy of incident nucleons, inelasticity, multiplicity and angular distribution vary event by event. This situation suggests that there is at least one more parameter which decides the details of the collision.

One of them could be, for example, the impact parameter. It seems difficult, however, to get the information of the magnitude of impact parameter from the experimental observation. Therefore, instead of directly approaching the problem of the impact parameter, we should like to take another way.

Let us devide the collision into two parts belonging to each nucleon. If we assume symmetry about the two colliding nucleons in C.M.S., the energy-momentum vector relations before and after collision are expressed as follows in the one dimensional case,

$$\begin{cases} E^*_{0f} = E^*_{Nf} + E^*_{Mf} \\ P^*_{0f} = P^*_{Nf} + P^*_{Mf} + \Delta P^* \end{cases} \tag{3.1}$$

$$\begin{cases} E^*_{0b} = E^*_{Nb} + E^*_{Mb} \\ -P^*_{0b} = -P^*_{Nb} - P^*_{Mb} - \Delta P^* , \end{cases} \tag{3.2}$$

where

$$(E^*_{0,N,M})_f = (E^*_{0,N,M})_b ,$$
$$(P^*_{0,N,M})_f = -(P^*_{0,N,M})_b .$$

The correction term $\pm \Delta P^*$ is introduced in consequence of the division of the collision into two parts (*). If we regard (3.1) or (3.2) as the conservation equation of the energy-momentum vector before and after collision in each part, this correction term must have the properties of a 4-vector, $(\pm \Delta P^*, 0)$ whose fourth component is equal to zero, and can be regarded as the parameter which represents the magnitude of the effect of the collision exerted by the partner as well as that exerted on the partner. Let us distinguish it from the energy-momentum vector of the real particle, calling it as the « momentum of interaction ».

The collision in L.S. is expressed as follows from the relations (3.1) and (3.2)

$$\begin{cases} E_{0f} = E_{Nf} + E_{Mf} + \gamma^*_0 \beta^*_0 \Delta P^* \\ P_{0f} = P_{Nf} + P_{Mf} + \gamma^*_0 \Delta P^* \end{cases} \tag{3.3}$$

$$\begin{cases} E_{0b} = 1 = E_{Nb} + E_{Mb} - \gamma^*_0 \beta^*_0 \Delta P^* \\ P_{0b} = 0 = P_{Nb} + P_{Mb} - \gamma^*_0 \Delta P^* . \end{cases} \tag{3.4}$$

(*) In the case of an asymmetric collision it must be $(E^*_{N,M})_f \neq (E^*_{N,M})_b$, and the fourth component of the correction term must have finite value. As the first step of our analisis, however, we confine ourselves to the symmetric case.

Then the « momentum of interaction » must be transformed from C.M.S. to L.S. as follows

$$(3.5)\qquad \begin{cases} (+\Delta P^*,\ 0,\ 0,\ 0) & (+\gamma_0^*\Delta P^*,\ 0,\ 0,\ +\gamma_0^*\beta_0^*\Delta P^*) \\ (-\Delta P^*,\ 0,\ 0,\ 0) & (-\gamma_0^*\Delta P^*,\ 0,\ 0,\ -\gamma_0^*\beta_0^*\Delta P^*)\,. \end{cases}$$

3·2. *Experimental distribution of the magnitude of the « momentum of interaction ».* – Let us study the experimental distribution of the magnitude of the « momentum of interaction » before investigating its physical meaning. From (3.1) and (3.2) following approximate relation is obtained, assuming small deflection of nucleons in the collision.

$$(3.6)\qquad \Delta P_{\parallel}^* \approx \frac{M_M}{2\gamma_M^*} \approx \frac{n_{f(b)}^2 P_{T\pi}^2}{2k^*\gamma_0^*}\,.$$

Using the relation (3.6), the value of $\Delta P_{\parallel}^*$ of each jet was calculated. In Fig. 6, these values are plotted versus γ_0^*. From this figure an interesting result is

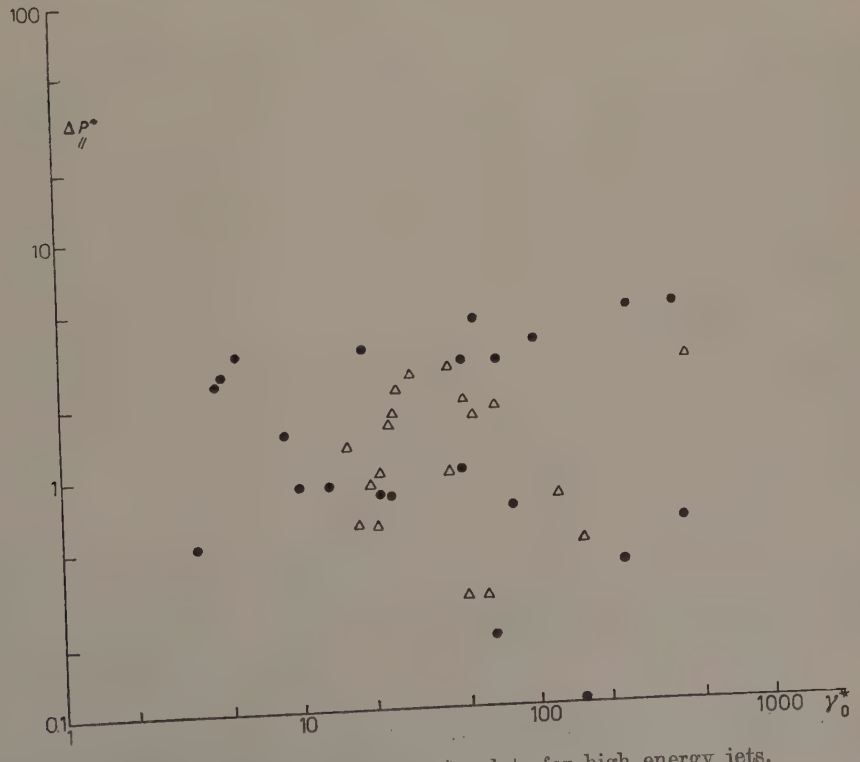


Fig. 6. – The $\Delta P_{\parallel}^*$ versus γ_0^* plots for high energy jets.

obtained. That is; the average value of $\Delta P_{\parallel}^{*}$ which is of the order of ~ 1 seems to be constant and independent from γ_0^*.

As for the $\Delta P_{\perp}$, it coincides with the transverse momentum of nucleons after the collision, because meson lumps travel along the incident direction of nucleons in our model and $P_{M\perp}^{*} = 0$. Judging from the report of the Bristol group that the transverse mnmentum of a particle which is not a π-meson is of the order of ~ 1, the order of $\Delta P_{\perp}$ cannot be much larger than 1, either. Actually, in the case of the G jet, it is about 0.5.

Then we can conclude that the average magnitude of the « momentum of interaction » in C.M.S. is of the order of 1 which is independent from the incident energy of a nucleon,

$$\langle \Delta P^{*} \rangle \approx 1 \,. \tag{3.7}$$

It is worth noticing that the « momentum of interaction » seems to be a quantity which is independent from the energy of an incident nucleon in C.M.S.

3·3. *Experimental relations among the « momentum of interaction » and other parameters.* – The experimental relations between the « momentum of interaction » and other parameters are shown as follows. The events adopted here are the same as those adopted in Fig. 6. In Fig. 7 and Fig. 8 the values M_M

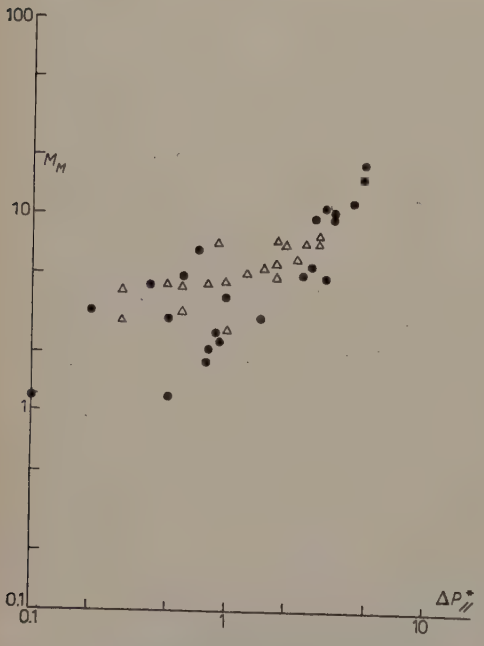


Fig. 7. – The M_M versus $\Delta P_{\parallel}^{*}$ plots for high energy jets.

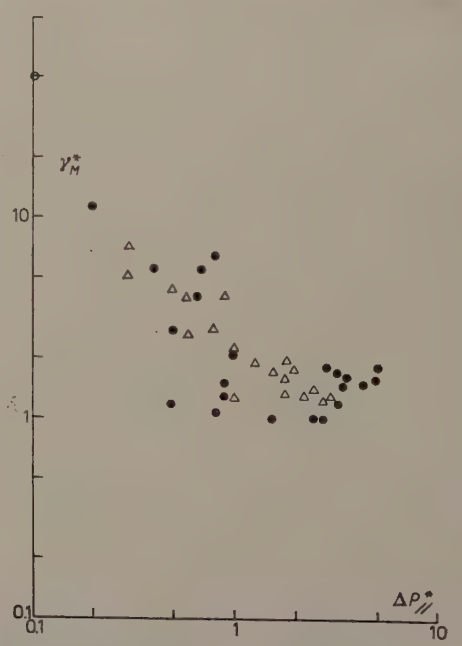


Fig. 8. – The γ_M^* versus $\Delta P_{\parallel}^{*}$ plots for high energy jets.

and γ_M^* of each jet are plotted versus ΔP^*. From the Fig. 7, M_M seems to be in proportion to $\sim \Delta P^{*\frac{1}{2}}$. As is expected from this relation and the relation (3.6), γ_M^* is nearly in proportion to $\sim \Delta P_{\parallel}^{*-\frac{1}{2}}$. It is very interesting that $\Delta P_{\parallel}^*$ seems to have such a simple relation to M_M or γ_M^* as shown above.

On the other hand, we cannot reach any conclusion about the relation between $\Delta P_{\parallel}^*$ and K^*. This situation is shown in Fig. 9, where K^* is plotted versus $\Delta P_{\parallel}^*$.

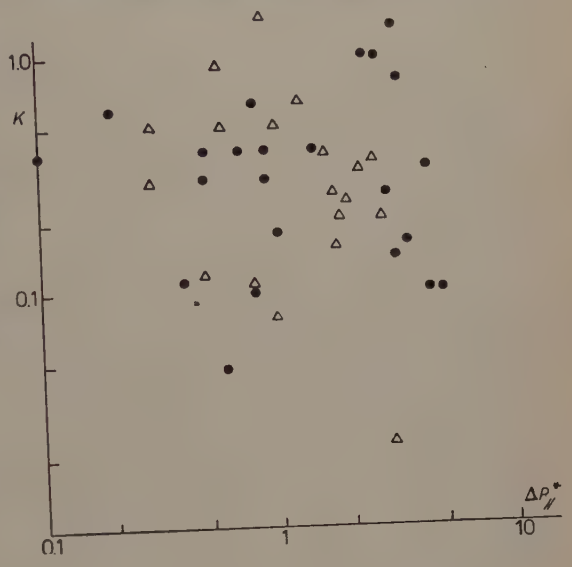


Fig. 9. – The K^* versus $\Delta P_{\parallel}^*$ plots for high energy jets.

3·4. *Discussions about the « momentum of interaction »*. – In this chapter the « momentum of interaction » has been introduced and its average value in C.M.S. has turned out to be of the order of ~ 1 which is independent from the incident energy of the nucleon, and moreover its relations to other parameters have been recognized. These circumstances seem to indicate the possibility of the « momentum of interaction » to be a parameter which decides the details of the collision.

Of course, there may be an alternative point of view that the important thing would be rather the correlation among the quantities $P_{\top\pi}$, n_f, K^*, and γ_0^* than ΔP^* itself, because $\Delta P_{\parallel}^*$ is the function of them as seen from the relation (3.6), and the constancy of $\Delta P_{\perp}^*$ has a close connection with the fact that the transverse momentum of a π-meson is small and constant. We should like, however, to take the former and to think that the « momentum of interaction » is an important quantity in the investigation of the dynamics of multiple meson production as the transverse momentum of mesons.

There are five independent quantities in the relation (3.1), and we have only two equations which relate them. Therefore, the whole aspects of the collision cannot be decided even when γ_0^* and ΔP^* are given. To describe the collision completely one more parameter, for example the degree of inelasticity, or one more equation relating the five quantities must be given.

When we investigate the problem of the inelasticity the persistency of the original nucleon throughout the collision should be taken into account. Ac-

cording to the empirical model indicated in Sect. 2 of this paper, we divide the collision phenomenon into two successive steps. In the first step colliding nucleons are decelerated and deflected by the mutual interaction and consequently two meson lumps are generated. In the second step the nucleons fly away along the direction ensuing from the first step and the meson lumps decay in flight in an extremely short time. Then the degree of inelasticity is determined in the first step of the collision.

Being regarded as a parameter representing the magnitude of the effect of the interaction between two nucleons, the « momentum of interaction » must play an essential role in the first step of the collision. If the « momentum of interaction » is assumed to play a role as an operator transforming a nucleon from the initial state to the final state, and a meson lump is assumed to be emitted so as to satisfy the principle of the energy and momentum conservation in the first step of the collision, the magnitude of the « momentum of interaction » must have a close connection with the difference vector between the initial and final state of the nucleon, $\Delta P_N^* = P_0^* - P_N^*$. In the high energy case the value $\Delta P_N^*/P_0^*$ is neary equal to K^*. Therefore, we have tryed to find some relation between ΔP_N^* and K^* but without success as was shown in Fig. 9.

In the above case, however, we have neglected the transverse component of the « momentum of interaction » whose average magnitude is expected to be of equal order of that of ΔP^*. Accordingly, even though a complicated and delicate relation exists between the magnitude of the « momentum of interaction » and the degree of inelasticity, it cannot possibly be found out in such a rough research.

Therefore, let us again come back to the G jet which is analysed in detail in the first part of this paper. Examining it in full detail, an unexpected strange situation was found. That is; in the G jet the magnitude of $\Delta P_{\parallel}^*$ seems to be equal to that of $\Delta P_{Nb\parallel}$

$$\text{(3.8)} \qquad \begin{cases} \Delta P_{\parallel}^* \approx 0.3 \\ \Delta P_{Nb\parallel} \approx 0.3 \\ \Delta P_{\parallel}^* \approx \Delta P_{Nb} . \end{cases}$$

Of course, this may be nothing other than a mere accident, because one of them is a quantity defined in the C.M.S. and the other is that defined in the L.S. and no direct connection between them is expected.

In reference to this situation, however, following circumstances should be noticed. The experimental information about the inelasticity was obtained

by various methods ([10]). And the average value of the inelasticity is $\approx 0.3 \pm 0.2$ for the energy region higher than 10^{11} eV. From this value we know that the average value of the recoil momentum of the nucleon in the L.S is of the order of ~ 1,

$$\langle \Delta P_{Nb} \rangle \approx 1 . \tag{3.9}$$

Comparing (3.9) and (3.7), it is not impossible to think that the relation (3.8) always helds in general nucleon-nucleon collisions and then ΔP^* is connected with K^* through this relation.

Needless to say, it is very strange to see such a simple direct correlation between the quantities defined in the C.M.S. and in the L.S.

At present, however, between the quantities in (3.1) we have no other equation which is necessary to describe the collision completely. Therefore, even if the relation (3.8) contradicts common sense at present, we should not throw it away immediately. To construct a complete model, in the next section we shall take the relation (3.8) as a working hypothesis saying nothing of its physical meaning.

4. - An empirical model of multiple meson production in high energy nucleon-nucleon collisions.

4·1. *Description of the model.* – In this Section an empirical model of multiple meson production in high energy nucleon-nucleon collisions will be constructed on the basis of the experimental results described in the preceding two sections.

First of all, let us assume that the following direct correlation between the « momentum of interaction », ΔP^*, which is defined in C.M.S. and the recoil momentum of the target nucleon in L.S., ΔP_{Nb}, always exists in every nucleon-nucleon collision,

$$\Delta P^* \equiv \Delta P_{Nb} . \tag{4.1}$$

Then, P_N^* should be given as follows in connection with ΔP^*,

$$\begin{cases} P_{N\parallel}^* = \gamma_0^* (-\Delta P_{\parallel}^* + \beta_0^* \sqrt{1 + \Delta P^{*2}}) , \\ P_{N\top}^* = \Delta P_{\top}^* . \end{cases} \tag{4.2}$$

([10]) K. Greisen and W. D. Walker: *Phys. Rev.*, **90**, 915 (1953); G. Bertolino and D. Pescetti: *Nuovo Cimento*, **12**, 630 (1954); G. Bertolino: *Nuovo Cimento*, **3**, 141 (1956); A. Engler, U. Haber-Schaim and W. Winkler: *Nuovo Cimento*, **12**, 930 (1954); P. Caldirola, R. Fieschi and P. Gulmanelli: *Nuovo Cimento*, **9**, 5 (1952); G. T. Zacepin: Oxford Conference Note, 8 (1956); E. Lohrman: *Nuovo Cimento*, **5**, 1074 (1957); N. M. Duller and W. D. Walker: *Phys. Rev.*, **93**, 215 (1954).

The relation (4.1) is realized by the transformation of (4.2) from C.M.S. to L.S.,

$$\text{(4.3)} \qquad \begin{cases} P_{Nb\parallel} = \gamma_0^*(-P_{N\parallel}^* + \beta_0^* E_N^*) = \Delta P_{\parallel}^* , \\ P_{NbT} = \Delta P_T^* . \end{cases}$$

The collision process will then be described as follows. Two nucleons encounter with equal momentum but opposite sign in the C.M.S. Then they are decelerated and deflected by the mutual interaction and consequently two meson lumps are emitted to satisfy the principle of the energy and momentum conservation in each nucleon side. In this first step of the collision the « momentum of interaction » plays two roles by itself; on one hand it decides the difference between the momenta of the initial state and the final state, which includes the final state of the nucleon and the generated meson lump, and on the other hand it determines the degree of inelasticity. In the second step the nucleon flies away as has been decided in the first step and the meson lump decays in flight in an extremely short instant emitting those mesons whose relative momenta to the center of the lump are, on an average, several hundred MeV/c. These collision processes are exhibited in Fig. 10, and Table II.

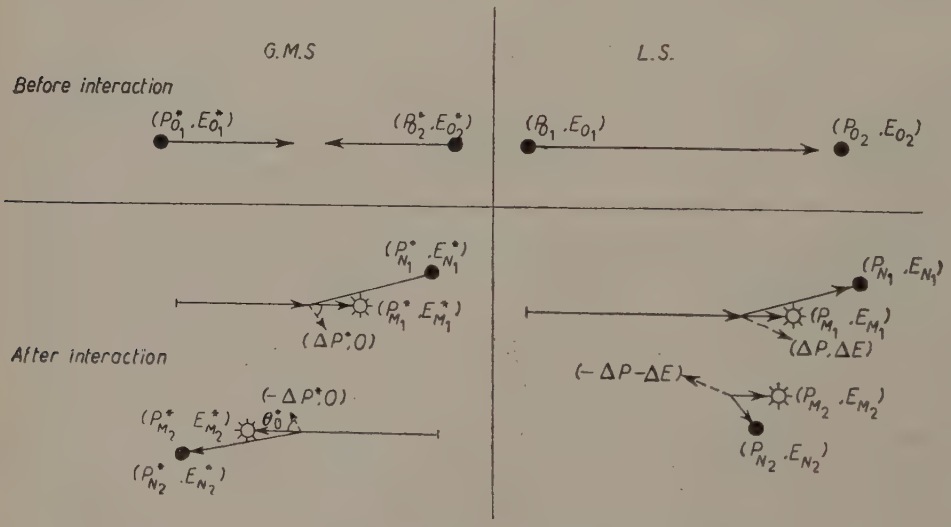


Fig. 10. – A schematic diagram of our empirical model.

Assumption (4.1) does not put a limitation on the possibility of realization of various degrees of inelasticity in the nucleon-nucleon collision, if we can take P_0^* as the maximum value of ΔP^* and 0 as the minimum one.

This is shown as follows.

TABLE II.

	C.M.S.		L.S.	
	Before collision	After collision	Before collision	After collision
Nucleon 1	$P^*_{01\parallel}$ $P^*_{01\perp}=0$ E^*_{01}	$P^*_{N1\parallel}=\gamma^*_0(-\Delta P^*_\parallel+\beta^*_0\sqrt{\Delta P^{*2}+1})$ $P^*_{N1\perp}=+\Delta P^*_\perp$ $E^*_{N1}=\sqrt{P^{*2}_{N1\parallel}+P^{*2}_{N1\perp}+1}$	P_{01} $P_{01\perp}=0$ E_{01}	$P_{N1\parallel}=\gamma_0(-\Delta P^*_\parallel+\beta_0\sqrt{\Delta P^{*2}+1})$ $P_{N1\perp}=+\Delta P^*_\perp$ $E_{N1}=\sqrt{P^2_{N1\parallel}+P^2_{N1\perp}+1}$
Meson lump 1	—	$P^*_{M1\parallel}=P^*_{01\parallel}-P^*_{N1\parallel}-\Delta P^*_\parallel$ $P^*_{M1\perp}=0$ $E^*_{M1}=E^*_{01}-E^*_{N1}$	—	$P_{M1\parallel}\ P_{01}\quad P_{N1\parallel}-\gamma^*_0\Delta P^*_\parallel$ $P_{M1\perp}=0$ $E_{M1}=E_{01}-E_{N1}-\gamma^*_0\beta^*_0\Delta P^*_\parallel$
Vector of interaction 1	—	$+\Delta P^*_\parallel$ $-\Delta P^*_\perp$ $\Delta E^*=0$	—	$\Delta P_\parallel=\gamma^*_0\Delta P^*_\parallel$ $\Delta P_\perp=-\Delta P^*_\perp$ $\Delta E=\gamma^*_0\beta^*_0\Delta P^*_\parallel$
Nucleon 2	$P^*_{02\parallel}=-P^*_{01\parallel}$ $P^*_{02\perp}=0$ $E^*_{02}=E^*_{01}$	$P^*_{N2\parallel}=\gamma^*_0(\Delta P^*_\parallel-\beta^*_0\sqrt{\Delta P^{*2}+1})$ $P^*_{N2\perp}=-\Delta P^*$ $E^*_{N2}=\sqrt{P^{*2}_{N2\parallel}+P^{*2}_{N2\perp}+1}$	$P_{02\parallel}=0$ $P_{02}=0$ $E_{02}=1$	$P_{N2\parallel}=\gamma_0(\Delta P^*_\parallel-\beta_0\sqrt{\Delta P^{*2}+1})=\Delta P^*_\parallel$ $P_{N2\perp}=-\Delta P^*_\perp$ $E_{N2}=\sqrt{P^2_{N2\parallel}+P^2_{N2\perp}+1}$
Meson lump 2	—	$P^*_{M2\parallel}=P^*_{02}-P^*_{N2\parallel}+\Delta P^*_\parallel$ $P^*_{M2\perp}=0$ $E^*_{M2}=E^*_{02}-E^*_{N2}$	—	$P_{M2}=P_{02}-P_{N2\parallel}+\gamma^*_0\Delta P^*_\parallel=\Delta P^*_\parallel(\gamma^*_0-1)$ $P_{M2\perp}=0$ $E_{M2}=E_{02}-E_{N2}+\gamma^*_0\beta^*_0\Delta P^*_\parallel$
Vector of interaction 2	—	$-\Delta P^*_\parallel$ $+\Delta P^*_\perp$ $\Delta E^*=0$	—	$\Delta P_\parallel=-\gamma^*_0\Delta P^*_\parallel$ $\Delta P_\perp=+\Delta P^*_\perp$ $\Delta E=-\gamma^*_0\beta^*_0\Delta P^*_\parallel$

When $\Delta P^* = P_0^*$, we get

$P_N^* = 0$ from (4.2)

and $P_M^* = 0$ from (3.1) and (3.2)

then $K^* = 1$.

When $\Delta P^* = 0$, we get

$P_N^* = P_0^*$ from (4.2)

and $P_M^* = 0$ from (3.1) and (3.2)

then $K^* = 0$.

Now we have got the empirical model of multiple meson production in which the variety of the final state of the collision can be derived from the initial conditions γ_0^* and $(\Delta \boldsymbol{P}^*, 0)$. In order to make the problem simple, at first, we should like to neglect the production of the K-mesons and nucleon-antinucleon pairs which must be included into the model suitably assuming the properties of meson lumps.

4·2. *The relations between characterizing parameters of the model and initial conditions.* – Next, let us study what kind of functions of the initial conditions are the characterizing parameters of the model.

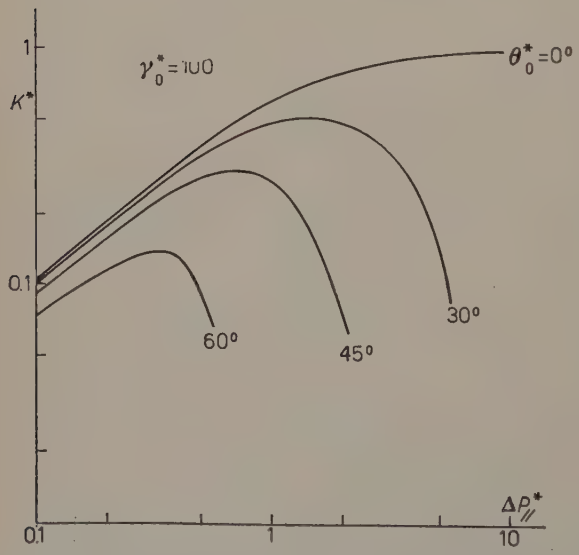


Fig. 1. – The K^* versus $\Delta P_{/\!/}^*$ curves for the fixed value of $\gamma_0^* = 100$.

4·2.1. The inelasticity of the collision, K^*. – The inelasticity of the collision is expressed as follows,

$$k^* = \frac{\gamma_0^* - \gamma_N^*}{\gamma_0^* - 1} = 1 - \frac{\gamma_N^* - 1}{\gamma_0^* - 1} .$$

In the high energy region we get the following approximate relation from (4.1)

$$k^* \approx 1 - \frac{\gamma_N^*}{\gamma_0^*} =$$

$$= 1 - \frac{\gamma_0^* (\sqrt{1 + \Delta P^{*2}} - \beta_0^* P_{/\!/}^*)}{\gamma_0^*} \approx$$

$$\approx 1 - \sqrt{1 + \Delta P^{*2}} + \Delta P^* \cos \theta_0^* ,$$

where

$$\theta_0^* = \operatorname{arctg} \frac{\Delta P_\perp}{\Delta P_\parallel}$$

Therefore, when we restrict ourselves to the high energy region the inelasticity becomes independent from the incident energy of a nucleon, and is given as a function of ΔP^* and θ_0^*. The relation between K^* and $\Delta P_\parallel^*$ is shown in Fig. 11. If we fix the values of ΔP^*, for example $\Delta P^* = 0.5$, K^* is given as a function of γ_0^* and θ_0^* as is shown in Fig. 12.

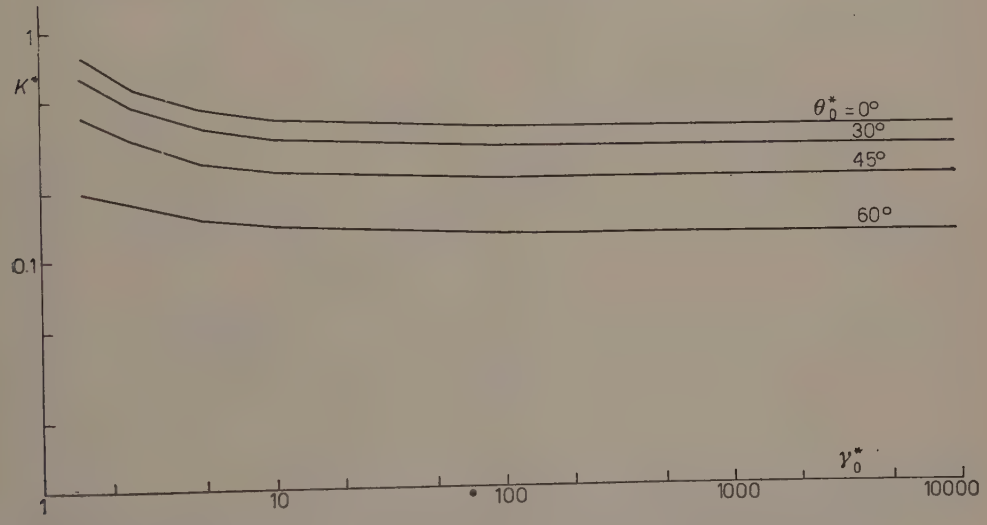


Fig. 12. – The K^* versus γ_0^* curves for the fixed value of $\Delta P^* = 0.5$.

4·2.2. The mass of the meson lump, M_M. – In our model the mass of the meson lump is given as follows,

$$M_M^2 = E_M^{*2} - P_M^{*2} = (E_0^* - E_N^*)^2 - (P_0^* - P_N^* - \Delta P^*)^2 =$$
$$= 2\Delta P^*(P_0^* - P_N^*) + 2 - 2(E_0^* E_N^* - P_0^* P_N^*) - \Delta P^{*2}.$$

If the incident momentum of the nucleon is much higher than ΔP^*, we get approximately

$$M_M^2 \approx 2\Delta P^*(P_0^* - P_N^*) \approx 2\Delta P^* \gamma^*(\beta^* + \Delta P^* - \beta^* \sqrt{1 + \Delta P^{*2}}).$$

Fixing the value of ΔP^*, we get an approximate relation between M_M and γ_0^*.

$$M_M \propto \gamma_0^{*\frac{1}{2}} \propto \gamma_0^{\frac{1}{2}}.$$

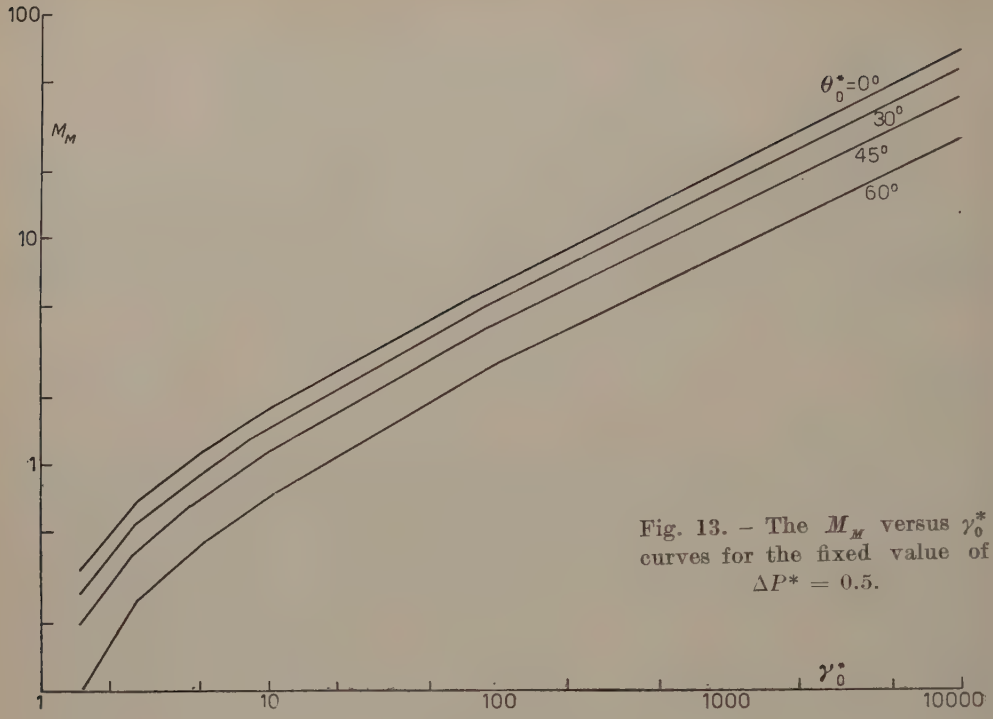
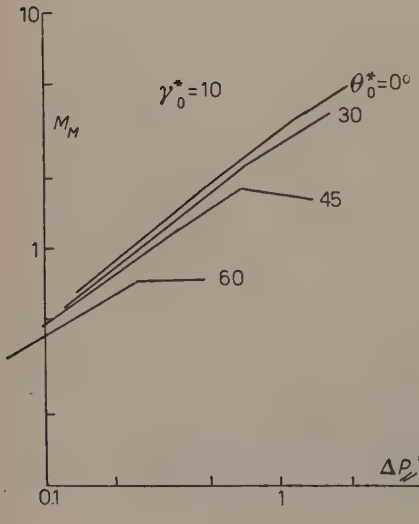
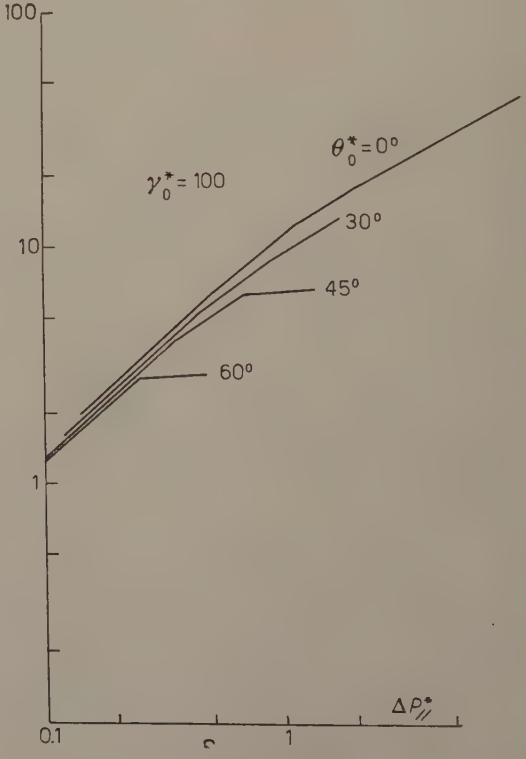


Fig. 13. – The M_M versus γ_0^* curves for the fixed value of $\Delta P^* = 0.5$.

Fig. 14. – The M_M versus $\Delta P_{\parallel}^*$ curves for two fixed values of $\gamma_0 = 10$ and 100.





On other hand we get the next relation in the low energy region,

$$M_M \propto \gamma_0^* \propto \gamma_0^{*\frac{1}{2}} .$$

M_M is plotted versus γ_0^* in Fig. 13, where ΔP^* is fixed to 0.5. In Fig. 14 the

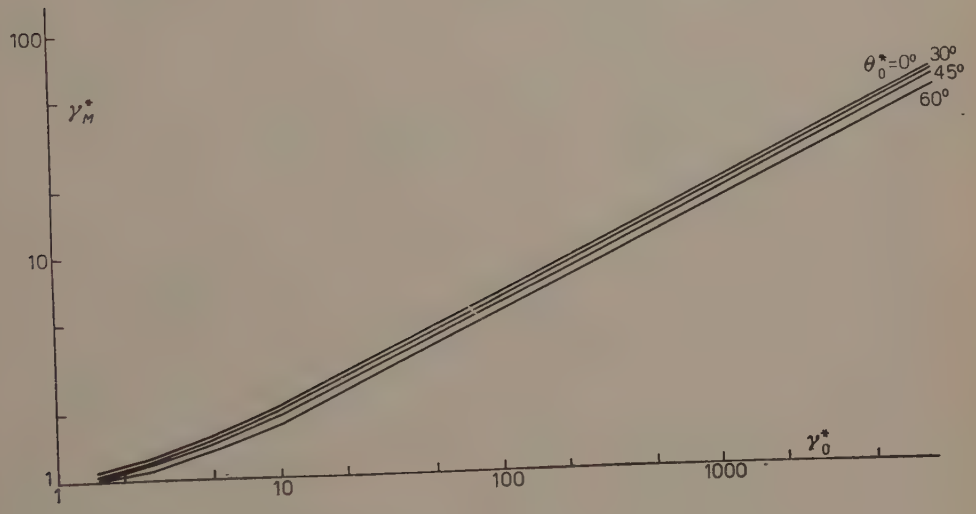


Fig. 15. – The γ_M^* versus γ_0^* curves for the fixed value of $\Delta P^* = 0.5$.

relation between M_M and ΔP^* is shown for the fixed value of γ_0^*. As is shown in this figure M_M is nearly in proportion to $\Delta P^{*\frac{1}{2}}$ free from γ_0^*.

4·2.3. The Lorentz factor of the meson lump, γ_M^*. – The Lorentz factor of the meson lump in the C.M.S. is nearly in proportion to $\gamma_0^{*\frac{1}{2}}$ when ΔP^* is fixed in our model. This is shown in Fig. 15 where ΔP^* is fixed to 0.5. The relation between γ_M^* and ΔP^* is shown in Fig. 16 for the fixed value of γ_0^*.

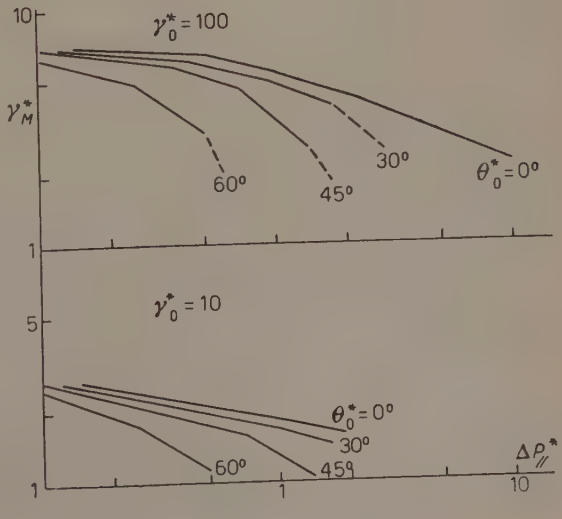


Fig. 16. – The γ_M^* versus $\Delta P_{\parallel}^*$ curves for two fixed values of $\gamma_0^* = 10$ and 100.

4·3. *The correlations between the characterizing parameters of our model.* – The correlations among the characterizing parameters of the model are

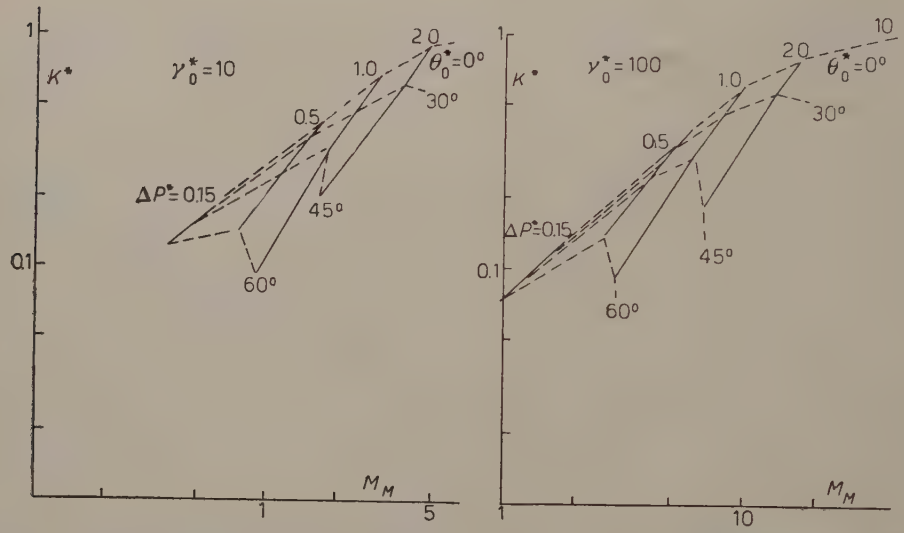


Fig. 17. – The relation between K^* and M_M for two fixed values of $\gamma_0^* = 10$ and 100.

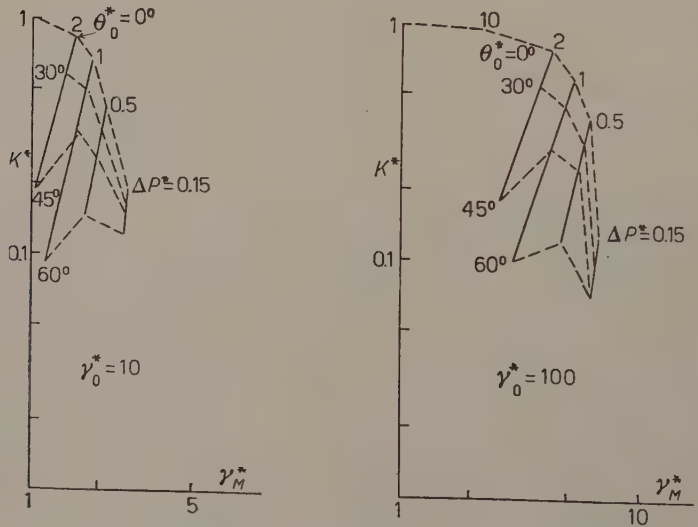


Fig. 18. – The relation between K^* and γ_M^* for two fixed values of $\gamma_0^* = 10$ and 100.

shown in Figs. 17, 18 and 19, for two fixed values of γ_0^*, 10 and 100. Even though we restrict ourselves to a fixed value of γ_0^*, the correlations

among these parameters are very complicated and the complication can be seen at a glance of these figures.

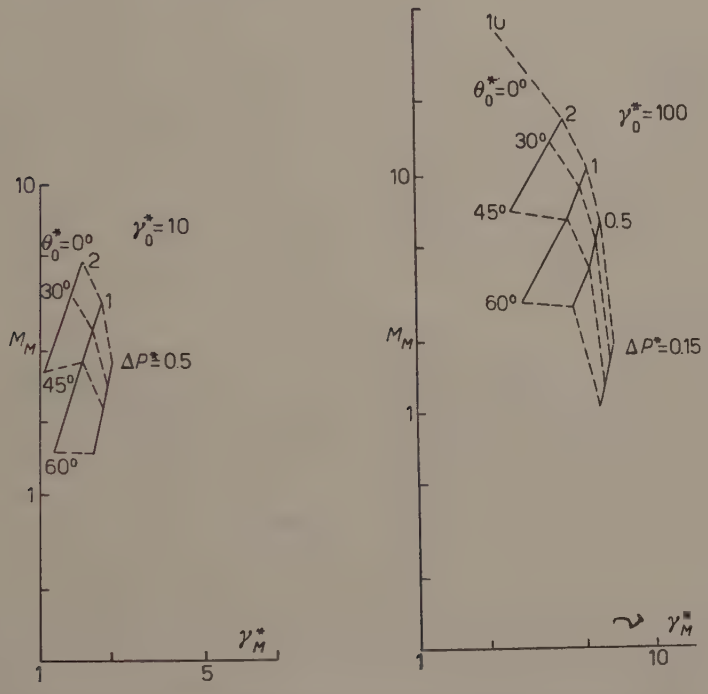


Fig. 19. – The relation between M_M and γ_M^* for two fixed values of $\gamma_0^* = 10$ and 100.

4·4. *Preliminary comparison of our model with the experimental data.* – Here we should like to restrict ourselves to the following qualitative comparison between our model and experimental results.

4·4.1. The relations of the characterizing parameters to the «momentum of interaction». – The experimental relations of M_M and γ_M^* to $\Delta P_\perp^*$ which are shown in Fig.s 7 and 8 are compared with those derived from our model which are shown in Figs. 14 and 16. Both experimental and calculated results show that M_M and γ_M^* are nearly in proportion to $\Delta P^{*\frac{1}{2}}$ and $\Delta P^{*-\frac{1}{2}}$ respectively. In our model K^* does not depend on $\Delta P_\perp^*$ as is shown in Fig. 11. This relation between K^* and $\Delta P_\perp^*$ is also compared with the experimental results shown in Fig. 9. From the comparison of these figures it is clear that our model can explain these experimental features well.

4·4.2. The relation between K^* and M_M. – As for the relation between K^* and M_M, the Bristol group got the result that at a fixed energy, K^*

does not strongly depend on n_s. On the other hand, KANEKO and OKAZAKI [11] reported that n_s is always in proportion to K^*. In the case of our model, generally speaking, K^* does not depend on M_M either as is shown in Fig. 17. If we fix ΔP^* at the value $0.5 \sim 1.0$, however, K^* becomes nearly in proportion to M_M. Therefore, the relation between K^* and M_M depends on ΔP^* and the distribution of the value of ΔP^* must be taken into account in the discussion of this problem. From the comparison of the experimental data with the prediction of our model about the relation between K^* and M_M, it seems that on the average K^* is nearly in proportion to M_M as KANEKO and OKAZAKI assert, but the fluctuation is very large and the conclusion of the Bristol group emphasizes this large fluctuation.

5. – Discussion.

In Sect. 2 we considered the meson lump as a real object, but it could be regarded as a virtual one. It must be pointed out, however, that the average value of the relative momentum of the π-mesons to the center of the lump coincides with that of the momentum of the π-mesons produced in the proton-antiproton annihilation at rest. This indicates that the meson lumps which are generated in high energy nuclear interaction have the same nature as, but different mass from, those which are produced in the proton-antiproton annihilation at rest. The characteristics of the meson lump, the constancy of the average momentum of the π-mesons in the lump which is independent from the mass of the lump, the magnitude of the average momentum of the K-mesons in the lump, the dependency of the number of K-mesons on the mass of the lump, etc., must be investigated in detail in future.

As was shown in Sect. 2, the nucleon after the collision cannot be the center of the meson lump, because, in general, its Lorentz factor differs widely from that of the meson lump, and this is the basis of our insistence that the original nucleon maintains its individuality and a good fraction of its energy throughout the interaction. In the statistical models of multiple meson production in the nuclear interaction, including Takagi's [1] and Kraushaar-Mark's [1] models, the original nucleons are buried in the excited bodies which are generated by the collisions and in which a statistical equilibrium is postulated for all particles. Therefore, the above situation cannot be explained by these models.

The « momentum of interaction » was introduced in Sect. 3 in consequence of the division of the collision into two parts belonging to each nucleon. It is the momentum difference between the initial state and the final state of each part in the C.M.S. Taking into account the persistency of the original nucleon

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[11] S. KANEKO and S. OKAZAKI: *Soryūsiron Kenkyū*, **16**, 231 (1958) in Japanese.

throughout the interaction, it can be regarded as a parameter which represents the magnitude of the effect of the interaction exerted by the partner as well as that exerted on the partner. Then it must play an essential role at the first step of the collision where the degree of the inelasticity is decided.

Introducing the « momentum of interaction » and using its empirical relation to the recoil vector of a nucleon described in Sect. 3, our model of multiple meson production in the nucleon-nucleon collision gets the possibility to explain the variety of the experimental data of high energy jets as is seen from its preliminary comparison with the experimental data. In order to consolidate the experimental foundation of our model, however, it is necessary to accumulate and analyse much more jets which probably correspond to nucleon-nucleon collision.

When we analyse the jet showers in the photographic emulsions, the energy and momentum of the recoil nucleon in the L.S. should be carefully investigated, because we can get the information of the inelasticity from their approximate relation to the inelasticity which is free from the incident energy as well as from the type of the model and which is shown as follows.

$$1 - K^* = \frac{E_N^* - 1}{E_0^* - 1} \approx \frac{E_N^*}{E_0^*} = \frac{\gamma_0^* (E_{Nb} - \beta_0^* P_{Nb\parallel})}{\gamma_0^*} \approx E_{Nb} - P_{Nb\parallel} .$$

In the case of our model the momentum of the recoil nucleon in the L.S. is directly connected with the momentum component of the « momentum of interaction » in the C.M.S.

Therefore, if the applicability of our model to a general nucleon-nucleon collision is proved by the analysis of the experimental data which will be accumulated hereafter, we can get the information about the distribution of the magnitude of the « momentum of interaction » in the C.M.S. from that of the recoil vector of the nucleon in the L.S. Then we shall be able to approach the problem of the physical significance of this quantity in connection with the structure of the nucleon or the structure of the interaction of the nucleons.

The collision between a π-meson and a nucleon can also be treated in a similar way to that developed in this paper for nucleon-nucleon collisions. In the center of the momentum system of a π-meson-nucleon collision, the generation of two meson lumps and the persistency of the incident π-meson are assumed. The « momentum of interaction » is also defined. And the final state of the incident π-meson is decided in connection with the « momentum of interaction » by a relation similar to (4.2).

$$P_\pi^* = \gamma_{0\pi}^* (- \Delta P^* + \beta_{0\pi}^* \sqrt{\mu^2 + \Delta P^{*2}}) ,$$

where $\gamma_{0\pi}^*$: the Lorentz factor of the incident π-meson in the C.M.S.

μ : the rest mass of a π-meson in nucleon mass units.

The behaviour of the target nucleon during the collision is assumed to be just the same as that in the case of a nucleon-nucleon collision. Then the final results of the collision are derived from the initial conditions P_0^* and $(\Delta P^*, 0)$. The main difference between the nucleon-nucleon collision and the π-meson-nucleon collision in our model appears in the inelasticity as is shown in Table III. In this table the results of the π-meson-nucleon collision and

TABLE III.

	P_0^*	$P_{\Lambda 0\pi}^*$	M_M	γ_M^*	K^*	K
$P_0^* = 9.95$ $\quad \Delta P^* = 0.5$ $\quad \theta_0^* = 0°$						
Nucleon 1	9.95	6.12	1.81	2.09	0.42	~ 0.42
Nucleon 2	9.95	6.12	1.81	2.09	0.42	
π-meson	9.95	1.46	2.84	3.00	0.87	~ 0.87
Nucleon	9.95	6.12	1.81	2.09	0.42	
$P_0^* = 99.995$ $\quad \Delta P^* = 0.5$ $\quad \theta_0^* = 0°$						
Nucleon 1	99.995	61.8	6.13	6.23	0.39	~ 0.39
Nucleon 2	99.995	61.8	6.13	6.23	0.39	
π-meson	99.995	14.66	9.22	9.25	0.85	~ 0.85
Nucleon	99.995	61.8	6.13	6.23	0.39	

the nucleon-nucleon collision are given for the initial conditions $P_0^* = 9.95$ and 99.995, $\Delta P^* = 0.5$ and $\theta_0^* = 0°$. The inelasticity in the case of the π-meson-nucleon collision is always large and approximate to 1. This fact is compared with the experimental results of the Bristol group that the inelasticity of the secondary jets which is considered to be produced mostly by π-mesons is of the order of unity. The qualitative agreement between the calculated and observed inelasticity in the π-meson-nucleon collision suggests that our model has a wider field of application.

* * *

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RIASSUNTO (*)

Si sono trovate la struttura in due pezzi dei mesoni prodotti e la persistenza dei nucleoni originali sulle interazioni nucleari ad alta energia. Su tali basi sperimentali si costruisce un modello empirico della produzione multipla di mesoni in collisioni nucleone-nucleone. Assumendo come simmetrici fra di loro i due nucleoni che collidono nel C.M.S. la collisione si divide in due parti appartenenti ad ogni nucleone. Si introduce una grandezza fisica detta « impulso d'interazione » corrispondente alla differenza d'impulso tra gli stati iniziale e finale di ogni parte. Tale grandezza si considera come un parametro che rappresenta la grandezza dell'effetto dell'interazione nella collisione. Si trova che nei getti ad alta energia il valore medio di questa grandezza è dell'ordine dell'unità in unità di masse nucleoniche indipendentemente dalla energia incidente. Assumendo come ipotesi di lavoro una relazione empirica tra « l'impulso d'interazione » e il momento di rinculo del nucleone bersaglio nel S.L., quale è suggerito dall'analisi di un getto di alta energia, i dettagli della collisione sono determinati nel nostro modello dall'entità dell'energia incidente e da quella dell'« impulso d'interazione ». Questo modello è in grado di dar ragione della varietà dei dati sperimentali sui getti di alta energia raccolti finora. Il significato fisico delle ipotesi fondamentali di questo modello dovrà essere esaminato in futuro.

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Conditions of Superfluidity of Nuclear Matter.

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Summary. — The methods developed in the theory of superconductivity are applied to the study of the properties of nuclear matter. The interaction that brings to superfluidity of nuclear matter is considered to be weak. Nucleon-nucleon interaction is taken in the most general form. Asymptotic solutions of the system of equations are found, which allow one to obtain the conditions of superfluidity of the nuclear matter. The conditions of superfluidity are in general equivalent to the requirement of predominance of attraction in nucleon-nucleon potentials at the Fermi surface energy. The data on nucleon-nucleon potentials allow one to conclude that the conditions of superfluidity of nuclear matter are fulfilled.

1. – Introduction.

The concept of « nuclear matter » is used when studying the properties of the atomic nucleus. It is believed that apart from Coulomb repulsions between protons a system of a very large number, say A, of nucleons composed of Z protons and $A - Z$ neutrons with $Z \sim A/2$ in its lowest energy state forms a stable configuration which is called nuclear matter. The inner part of medium and heavy nuclei are samples of such nuclear matter. The dynamic properties of nuclear matter can be described in first approximation by an independent particle model.

The structure of nuclear matter is in many respects similar to the electron structure of metals. Therefore the mathematical methods successfully developed in the theory of superconductivity [1] are worth to be applied to the study of nuclear matter properties.

N. N. Bogoljubov [2] drew attention to the fact that nuclear matter can possess the properties of superfluidity. In the case of the Fermi-gas the system

[1] N. N. Bogoljubov, V. V. Tolmačev and D. V. Širkov: *New Method in the Theory of Superconductivity* (Moscow, 1958).

[2] N. N. Bogoljubov: *Docl. Akad. Nauk. SSSR*, **119**, 52 (1958).

possessing the lowest energy is that, in which the particles successively occupy all the energy levels up to some energy $E(k_F)$, the energy of the Fermi-surface, which depends on the density of the gas. This state of nuclear matter with the energy levels successfully occupied is called *normal*. Owing to the interaction of fermion pairs possessing equal and opposite momenta, the system of Fermions may acquire the properties of superfluidity. This means that the system of Fermions will have an energy state lower than the normal one. Such a state of nuclear matter is called *superfluid*.

Very weak interactions of electron pairs possessing equal and opposite momenta lead to the superfluid state of a solid. Superfluidity is connected with the regulation of the movement of the electrons near the Fermi surface. The mathematical methods developed in [1] are strictly connected with the weakness of these interactions.

It is known that the interaction between free nucleons is a strong one. Nevertheless, the interaction between nucleon pairs possessing equal and opposite momenta leading to the superfluid state of nuclear matter can be considered a weak one. Indeed, the successful application of the nuclear shell model shows that in first approximation the interaction forces between the nucleons may be reduced to a certain effective potential which does not affect the appearance of the superfluid state and to the change of the nucleon mass. Furtermore the interaction between nucleons, which leads to the superfluid state, depends on the degree of regularity in the movement of the nucleons which may be not very large.

In the present paper we shan't fortell which of the forces, central or non-central, singlet or triplet, proton-proton or proton-neutron must be responsible for the appearance of the superfluid state of nuclear matter. The aim of the present paper is to obtain the conditions which are to be satisfied by the most general form of potential of the nucleon-nucleon interaction in order to give rise the superfluid state of nuclear matter.

2. – The interaction Hamiltonian.

Let us consider the part of the Hamiltonian of the interaction which is responsible for the superfluid state of nuclear matter. We suppose that as in the theory of superconductivity, the interactions of fermion pairs possessing equal and opposite momenta lead to the superfluid state. The interactions of nucleon pairs possessing arbitrary momenta lead to some effective potential of nuclear matter which we shall not consider.

Let us write the interaction Hamiltonian of our model in the following way:

$$H_{\text{int}} = \frac{1}{V}\sum_{k,k'} \psi_1^*(k)\,\psi_2^*(-k)\,V(k, k')\,\psi_2(-k')\,\psi_1(k')\,, \tag{1}$$

where v is the volume of the system and $\psi_1(k)$, $\psi_2(k)$ are composed of the operators of production and absorption of protons and neutrons. Let us take the interaction $V(k, k')$ between two nucleons in the most general form:

$$
\begin{aligned}
(2)\qquad V(k,k') = {} & J_{ss}(k,k')\tfrac{1}{4}(1-\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\tfrac{1}{4}(1-\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + \\
& + J_{st}(k,k')\tfrac{1}{4}(1-\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\tfrac{1}{4}(3+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + \\
& + J_{ts}(k,k')\tfrac{1}{4}(3+\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\tfrac{1}{4}(1-\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + \\
& + J_{tt}(k,k')\tfrac{1}{4}(3+\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\tfrac{1}{4}(3+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) - \\
& - J_{sT}(k,k')\tfrac{1}{4}(1-\boldsymbol{\tau}_1\boldsymbol{\tau}_2)[3(\boldsymbol{\sigma}_1\boldsymbol{e})(\boldsymbol{\sigma}_2\boldsymbol{e}) - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 e^2] - \\
& - J_{tT}(k,k')\tfrac{1}{4}(3+\boldsymbol{\tau}_1\boldsymbol{\tau}_2)[3(\boldsymbol{\sigma}_1\boldsymbol{e})(\boldsymbol{\sigma}_2\boldsymbol{e}) - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 e^2] + \\
& + J_{sl}(k,k')\tfrac{1}{4}(1-\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2,\ \boldsymbol{l}) + \\
& + J_{tl}(k,k')\tfrac{1}{4}(3+\boldsymbol{\tau}_1\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2,\ \boldsymbol{l})\,.
\end{aligned}
$$

Here $l = (\boldsymbol{k}-\boldsymbol{k}')/(2\,|\,\boldsymbol{k}\,|)$, $\boldsymbol{l} = (\boldsymbol{k}\times\boldsymbol{k}')/(2\,|\,\boldsymbol{k}\,|)$, the first index s or t in the functions $J_{in}(k, k')$ means singlet or triplet states in the isotopic spin space, the second index refers to the usual spin space, s means the singlet interaction, t means a triplet one, T is interaction of tensor type, and l is the interaction of spin-orbit type.

From the Hermitian conjugation and from the invariance with respect to the operation of reflection of space co-ordinates we find that the functions $J_{in}(k, k')$ are real and possess the following properties:

$$
(3)\qquad J_{in}(k,k') = J_{in}(k',k)\,, \qquad J_{in}(k,k') = J_{in}(-k,-k')\,.
$$

Let us express the interaction Hamiltonian (1) in terms of operators of production and absorption of neutrons and protons, that is:

$$
\begin{aligned}
(4)\qquad H_{\text{int}} = \frac{1}{V}\sum_{k,k'}\Big\{ & J_N(k,k')[a^+_{k+}a^+_{-k-}a_{-k'-}a_{k'+} + b^+_{k+}b^+_{-k-}b_{-k'-}b_{k'+}] + \\
& + J_+(k,k')a^+_{k+}b^+_{-k'+}b_{-k'+}a_{k'+} + J_-(k,k')a^+_{-k-}b^+_{k-}b_{k'-}a_{-k'-} + \\
& + \tfrac{1}{2}J_v(k,k')[a^+_{k+}b^+_{-k-}b_{-k'-}a_{k'+} + a^+_{-k-}b^+_{k+}b_{k'+}a_{-k'-}] + \\
& + \tfrac{1}{2}J_\sigma(k,k')[a^+_{k+}b^+_{-k-}b_{k'+}a_{-k'-} + a^+_{-k-}b^+_{k+}b^+_{-k'-}a_{k'+}) + \\
& + J^{(1)}_{\text{I}}(k,k')a^+_{k+}b^+_{-k+}b_{-k'-}a_{k'+} + J^{(1)*}_{\text{I}}(k,k')a^+_{k+}b^+_{-k-}b_{-k'+}a_{k'+} + \\
& + J^{(2)}_{\text{I}}(k,k')a^+_{k+}b^+_{-k+}b_{k'+}a_{-k'-} + J^{(2)*}_{\text{I}}(k,k')a^+_{-k-}b^+_{k+}b_{-k'+}a_{k'+} + \\
& + J^{(3)}_{\text{I}}(k,k')a^+_{k+}b^+_{-k-}b_{k'-}a_{-k'-} + J^{(3)*}_{\text{I}}(k,k')a^+_{-k-}b^+_{k-}b_{-k'-}a_{k'+} + \\
& + J^{(4)}_{\text{I}}(k,k')a^+_{-k-}b^+_{k+}b_{k'-}a_{-k'-} + J^{(4)*}_{\text{I}}(k,k')a^+_{-k-}b^+_{k-}b_{k'+}a_{-k'-} + \\
& + J_{\text{II}}(k,k')a^+_{k+}b^+_{-k+}b_{k'-}a_{-k'-} + J^*_{\text{II}}(k,k')a^+_{-k-}b^+_{k-}b_{-k'+}a_{k'+}\Big\}\,.
\end{aligned}
$$

Here $a^+_{k\pm}$, $b^+_{k\pm}$ are the operators of proton and neutron production, $a_{k\pm}$, $b_{k\pm}$ are their absorption operators, and $\pm$ characterizes the spin direction. The functions $J_N(k, k')$, $J_+(k, k')$ and the other ones are to be expressed in terms of the functions $J_{\text{in}}(k, k')$.

Thus the Hamiltonian of our model will be:

$$(5) \qquad H = \sum_{k,s'} \{E(k) - E_F\} c^+_{ks} c_{ks} + \frac{1}{V} \sum_{\substack{k,k' \\ s_1 s_2 s'_1 s'_2}} I(k, k' | s_1, s_2; s'_2, s'_1) c^+_{ks_1} c^+_{ks_2} c_{k's'_2} c_{k's'_1} ,$$

where E_F is a parameter which plays the role of a chemical potential; in the normal state it is equal to the energy of the Fermi surface; s is a discrete index which characterizes spin and isotopic spin and the sign of the momentum k.

3. – The system of fundamental equations.

Below we follow the considerations of BOGOLJUBOV's paper (2). Let us perform the canonical transformation

$$(6) \qquad a_{ks} = \sum_{s'} \{u(k, s, s') \alpha_{ks'} + v(k, s, s') \alpha^+_{ks'}\} ,$$

Let us determine the new vacuum state $\alpha_{ks} C_0 = 0$ and evaluate the average value $\overline{H} = \langle C^*_0 H C_0 \rangle$. In $\overline{H}$ let us neglect the terms which remain finite when $V \to \infty$. The functions $u(k, s, s')$, $v(k, s, s')$ are to be found by means of the requirement that $\overline{H}$ be an extremum while fulfilling the additional conditions. The equation obtained yields the solution that corresponds to the normal state $C^{(n)}$.

In order to obtain the conditions under which the energy of the normal state $C^{(n)}$ will not be minimal and therefore the superfluid state $C^{(s)}$ will appear, let us evaluate the second variation of $\overline{H}$. If the second variation of $\overline{H}$ over $u(k, s, s')$ and $v(k, s, s')$ in the presence of additional conditions is less than zero for the solution corresponding to the normal state $C^{(n)}$, this means that the superfluid state $C^{(s)}$ appears.

Thus we obtain that the superfluid state of nuclear matter exists only when there are solutions with negative eigenvalues E of the following set of equations:

$$(7) \qquad 2 |E(k) - E_F| \psi(k) + \frac{1}{V} \sum_{k'} J_N(k, k') \psi(k') = E \psi(k) ,$$

$$(8) \qquad 2 |E(k) - E_F| \varphi_+(k) + \frac{1}{V} \sum_{k'} \{J_+(k, k') \varphi_+(k') + J^{(1)}_1(k, k') \varphi_1(k') + J^{(2)}_1(k, k') \varphi_2(k') + J_{II}(k, k') \varphi_-(k')\} = E \varphi_+(k) ,$$

$$(9)\quad 2|E(k)-E_F|\varphi_-(k)+\frac{1}{V}\sum_{k'}\{J_-(k,k')\varphi_-(k')+J_{\mathrm{I}}^{(3)*}(k,k')\varphi_1(k')+$$
$$+J_{\mathrm{I}}^{(4)*}(k,k')\varphi_2(k')+J_{\mathrm{II}}^{*}(k,k')\varphi_+(k')\}=E\varphi_-(k)\,,$$

$$(10)\quad 2|E(k)-E_F|\varphi_1(k)+\frac{1}{V}\sum_{k'}\{\tfrac{1}{2}J_\nu(k,k')\varphi_1(k')+\tfrac{1}{2}J_\sigma(k,k')\varphi_2(k')+$$
$$+J_{\mathrm{I}}^{(1)*}(k,k')\varphi_+(k')+J_{\mathrm{I}}^{(3)}(k,k')\varphi_-(k')\}=E\varphi_1(k)\,,$$

$$(11)\quad 2|E(k)-E_F|\varphi_2(k)+\frac{1}{V}\sum_{k'}\{\tfrac{1}{2}J_\nu(k,k')\varphi_2(k')+\tfrac{1}{2}J_\sigma(k,k')\varphi_1(k')+$$
$$+J_{\mathrm{I}}^{(2)*}(k,k')\varphi_+(k')+J_{\mathrm{I}}^{(4)}(k,k')\varphi_-(k')\}=E\varphi_2(k)\,.$$

4. – The conditions of superfluidity in the case of p-p and n-n interactions.

Before proceeding to the investigation of the conditions of superfluidity of nuclear matter in the most general form, we shall consider two particular cases. The first one, when proton-proton and neutron-neutron interactions are considered and proton-neutron interactions are neglected. Evidently this case is the most interesting one, when passing from nuclear matter to the atomic nucleus.

Let us find for which $J_\nu(k,k')$ it is possible to solve the equation

$$(7)\qquad 2|E(k)-E_F|\psi(k)+\frac{1}{V}\sum_{k'}J_N(k,k')\psi(k')=E\psi(k)\,,$$

with negative eigenvalues $E=-2\delta$, $\delta>0$. With this aim we have investigated the asymptotic form of the solution of (7) when $J_\Lambda(k,k')$ tends to zero as well as E, which remains negative. Let

$$\psi(k)=\frac{\chi(k,\Omega)}{|E(k)-E_F|+\delta}\,.$$

Passing from the sum to an integral we get:

$$(12)\qquad 2\chi(k,\Omega)+\frac{1}{(2\pi)^3}\int \mathrm{d}\Omega'\int_0^\infty k'^2\,\mathrm{d}k'\,\frac{J_N(|k|,|k'|,\cos\alpha)}{|E(k')-E_F|+\delta}\,\chi(k',\Omega')=0\,,$$

where $\cosh\alpha=\cosh\vartheta\cosh\vartheta'+\sinh\vartheta\sin\vartheta'\cosh(\varphi-\varphi')$.

Taking into account that when $\delta\to 0$ the integral in (12) becomes logarith-

mically divergent near the Fermi surface let us consider the approximated equation

$$(13)\quad \chi(k,\Omega)+\ln\left(\frac{mE'(k_F)}{\delta}\right)\cdot\frac{k_F^2}{E'(k_F)}\frac{1}{(2\pi)^3}\int d\Omega' J_N(|k|,k_F,\cosh\alpha)\chi(k_F,\Omega')-$$

$$-\frac{1}{2(2\pi)^3}\int d\Omega'\int_0^\infty dk'\ln|k'-k_F|\frac{d}{dk'}\left[J_N(|k|,|k'|\cosh\alpha)\frac{k'^2}{E(k')}\chi(k',\Omega')\right]=0,$$

which for small $J_N(k,k')$ asymptotically coincides with (12). Let us introduce a certain function

$$f(k',\Omega')=\frac{\chi(k',\Omega')}{\chi(k'_F,\Omega)}\frac{1}{\ln(mE'(k_F)/\delta)},$$

and

$$f(k_F)=\frac{1}{\ln(mE'(k_F)/\delta)}>0.$$

Then the equation for $f(k,\Omega)$ is

$$(14)\quad f(k,\Omega)+\frac{k_F^2}{E'(k_F)}\frac{1}{(2\pi)^3}\int d\Omega' J_N(|k|,k_F,\cosh\alpha)\frac{\chi(k_F,\Omega')}{\chi(k_F,\Omega)}-$$

$$-\frac{1}{2}\frac{1}{(2\pi)^3}\int d\Omega'\int_0^\infty dk'\ln|k'-k_F|\cdot$$

$$\cdot\frac{d}{dk'}\left[J_N(|k|,|k'|,\cosh\alpha)\frac{k'^2}{E'(k')}f(k',\Omega')\cdot\chi(k_F,\Omega')/\chi(k_F,\Omega)\right]=0.$$

The solution of the equation (14) at $k=k_F$ since $J_N(k,k')$ tends to zero exists only when

$$(15)\quad \int d\Omega'\frac{\chi(k_F,\Omega')}{\chi(k_F,\Omega)}J_N(k_F,k_F,\cosh\alpha)<0.$$

Actually $f(k_F)>0$ and the last term in (14) has a value of higher order of smallness if for $J_N(k_F,|k|,\cosh\alpha)$ there doesn't exist any region in k'-space where it rapidly changes.

Let us expand in Legendre polynomials

$$J_{in}(k_F,k_F,\pm\cosh\vartheta)=\sum_{l=0}^{\infty}J_{in}^l(k_F)P_e(\pm\cosh\vartheta)$$

$$\chi(k_F,\cosh\vartheta)=\sum_{l=0}^{\infty}d^e(k_F)P_e(\cosh\vartheta)$$

and perform the corresponding integration. Then we obtain the conditions of superfluidity of nuclear matter

$$J_{ts}^{l=0}(k_F) < 0\,, \qquad J_{ts}^{l=2}(k_F) < 0\,, \tag{16}$$

$$J_{tt}^{l=1}(k_F) + 0.8 J_{tT}^{l=1}(k_F) < 0 \tag{17}$$

if each state with $l > 2$ contributes relatively little to the p-p interaction.

5. – The conditions of superfluidity the in case of central forces.

It is of interest to consider the case of central forces. We come to it if we suppose that tensor and spin-orbit forces tend to zero more rapidly than central forces.

In the case of central forces a set of equations (7)–(11) is reduced to three independent equations:

$$2\,|E(k) - E_F|\,\psi(k) + \frac{1}{V}\sum_{k'} J_N^0(k, k')\,\psi(k') = E\psi(k)\,, \tag{18}$$

$$2\,|E(k) - E_F|\,\varphi_+(k) + \frac{1}{V}\sum_{k'} J_+^0(k, k')\,\varphi_+(k') = E\varphi_+(k)\,, \tag{19}$$

$$2\,|E(k) - E_F|\,\varphi_1(k) + \frac{1}{2V}\sum_{k'}\left\{J_v^0(k, k') + J_\sigma^0(k, k')\right\}\varphi_1(k') = E\varphi_1(k)\,. \tag{20}$$

Following the considerations of the previous section we get the conditions of superfluidity of nuclear matter.

$$J_{ts}^{l=0}(k_F) < 0\,, \qquad J_{ts}^{l=2}(k_F) < 0\,, \qquad J_{ss}^{l=1}(k_F) < 0\,, \qquad J_{tt}^{l=1}(k_F) < 0 \tag{21}$$

if the contribution of the states with $l > 2$ is not large.

6. – The conditions of superfluidity in the general case.

Now let us investigate the restrictions on the functions $J_{in}(k, k')$ in order that the set of equations (7)–(11) may have solutions with negative eigenvalues E. With this aim we suppose

$$\varphi_\pm(k) = \frac{\theta_\pm(k, \Omega)}{|E(k) - E_F| + \delta}; \qquad \varphi_{1,2}(k) = \frac{\theta_{1,2}(k, \Omega)}{|E(k) - E_F| + \delta}\,.$$

As in Sect. 4 we go over to the approximated set of equations which for

small J_{in} asymptotically coincides with the set of equations (7)–(11). Then we introduce some unknown functions $f_i(k, \Omega)$ coinciding between themselves for $k = k_F$. Suppose that the functions $J_{\text{in}}(k_F, |k'|, \Omega')$ have no region in k'-space, where they change very rapidly. It is convenient to introduce the notation

$$J_{\text{in}}^{(\pm)} = \tfrac{1}{2}\{J_{\text{in}}(\boldsymbol{k}_F, \boldsymbol{k}'_F) \pm J_{\text{in}}(\boldsymbol{k}_F, -\boldsymbol{k}'_F)\}.$$

Then the conditions of superfluidity of nuclear matter for the most general case of nucleon-nucleon interaction are

$$\int \mathrm{d}\Omega' \frac{\chi(k_F, \Omega')}{\chi(k_F, \Omega)}[J_{ts}^{(+)} + J_{tt}^{(-)} + 2(3e_z^2 - e^2)J_{tT}^{(-)}] < 0\,, \tag{22}$$

$$\begin{aligned}\mathrm{Re}\int \mathrm{d}\Omega' \Big\{ & \frac{\theta_+(k_F, \Omega')}{\theta_-(k_F, \Omega)}[J_{st}^{(+)} + J_{tt}^{(-)} - (3e_z^2 - e^2)(J_{sT}^{(+)} + J_{tT}^{(-)}) + 2e_z(J_{sl}^{(+)} + J_{tl}^{(-)})] + \\ & + \frac{\theta_1(k_F, \Omega')}{\theta_+(k_F, \Omega)}[(l_x - il_y)(J_{sl}^{(+)} + J_{tl}^{(-)}) - 3e_z(e_x - ie_y)(J_{sT}^{(+)} + J_{tT}^{(-)})] + \\ & + \frac{\theta_2(k_F, \Omega')}{\theta_+(k_F, \Omega)}[(l_x - il_y)(J_{sl}^{(+)} - J_{tl}^{(-)}) - 3e_z(e_x - ie_y)(J_{sT}^{(+)} - J_{tT}^{(-)}) - \\ & - \frac{\theta_-(k_F, \Omega')}{\theta_+(k_F, \Omega)}(3e_x^2 - 3e_y^2 - 6ie_xe_y)[J_{sT}^{(+)} - J_{tT}^{(-)}]\Big\} < 0\,,\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{Re}\int \mathrm{d}\Omega' \Big\{ & \frac{\theta_-(k_F, \Omega')}{\theta(k, \Omega_F)}[J_{st}^{(+)} + J_{tt}^{(-)} - (3e_z^2 - e^2)(J_{sT}^{(+)} + J_{tT}^{(-)}) - 2l_z(J_{sl}^{(+)} + J_{tl}^{(-)})] + \\ & + \frac{\theta_1(k_F, \Omega')}{\theta_-(k_F, \Omega)}[(l_x + il_y)(J_{sl}^{(+)} - J_{tl}^{(-)}) + 3e_z(e_x + ie_y)(J_{sT}^{(+)} - J_{tT}^{(-)})] + \\ & + \frac{\theta_2(k_F, \Omega')}{\theta_-(k_F, \Omega)}[(l_x + il_y)(J_{sl}^{(+)} + J_{tl}^{(-)}) + 3e_z(e_x + ie_y)(J_{sT}^{(+)} + J_{tT}^{(-)})] - \\ & - \frac{\theta_+(k_F, \Omega')}{\theta_-(k_F, \Omega)}(3e_x^2 - 3e_y^2 + 6ie_xe_y)[J_{sT}^{(+)} - J_{tT}^{(-)}]\Big\} < 0\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\mathrm{Re}\int \mathrm{d}\Omega' \Big\{ & \frac{1}{2}\frac{\theta_1(k_E, \Omega')}{\theta_1(k_F, \Omega)}[J_{ss}^{(-)} + J_{st}^{(+)} + J_{ts}^{(+)} + J_{tt}^{(-)} + (3e_z^2 - e^2)(J_{sT}^{(+)} + J_{tT}^{(-)})] + \\ & + \frac{1}{2}\frac{\theta_2(k_F, \Omega')}{\theta_1(k_F, \Omega)}[J_{ss}^{(-)} + J_{st}^{(+)} - J_{ts}^{(+)} - J_{tt}^{(-)} + (3e_z^2 - e^2)(J_{sT}^{(+)} - J_{tT}^{(-)})] + \\ & + \frac{\theta_+(k_F, \Omega')}{\theta_1(k_F, \Omega)}[(l_x + il_y)(J_{sl}^{(+)} + J_{tl}^{(-)}) - 3e_z(e_x + ie_y)(J_{st}^{(+)} + J_{tT}^{(-)})] + \\ & + \frac{\theta_-(k_F, \Omega')}{\theta_1(k_F, \Omega)}[(l_x - il_y)(J_{sl}^{(+)} - J_{tl}^{(-)}) + 3e_z(e_x - ie_y)(J_{sT}^{(+)} - J_{tT}^{(-)})]\Big\} < 0\,,\end{aligned} \tag{25}$$

$$\text{(26)}\qquad \mathrm{Re}\int \mathrm{d}\Omega' \left\{ \int_{\theta_2(k_F,\,\Omega)}^{\theta_2(k_F,\,\Omega')} [J_{ss}^{(-)} + J_{st}^{(+)} + J_{ts}^{(+)} + J_{tt}^{(-)} + (3e_z^2 - e^2)(J_{sl}^{(+)} + J_{tT}^{(-)})] + \right.$$

$$+ \frac{1}{2}\frac{\theta_1(k_F,\,\Omega')}{\theta_2(k_F,\,\Omega)} [J_{ss}^{(-)} + J_{st}^{(+)} - J_{ts}^{(+)} - J_{tt}^{(-)} + (3e_z^2 - e^2)(J_{sT}^{(+)} - J_{tT}^{(-)})] +$$

$$+ \frac{\theta_+(k_F,\,\Omega')}{\theta_2(k_F,\,\Omega)} [(l_x + il_y)(J_{sl}^{(+)} - J_{te}^{(-)}) - 3e_z(e_x + ie_y)(J_{sT}^{(+)} - J_{sT}^{(-)})] +$$

$$\left. + \frac{\theta_-(k_F,\,\Omega')}{\theta_2(k_F,\,\Omega)} [(l_x - il_y)(J_{sl}^{(+)} + J_{te}^{(-)}) + 3e_z(e_x - ie_y)(J_{sT}^{(+)} + J_{tT}^{(-)})] \right\} < 0$$

and the imaginary parts of the expressions (23)–(26) must be of the same order of smallness as the terms neglected.

The conditions of superfluidity of nuclear matter (22)–(26) and the restrictions for the imaginary parts are very complicated. Nevertheless as it was shown in the previous sections, they give us the possibility to obtain simple conditions for the particular cases of interactions.

7. – Conclusion.

As shown by the theory of superconductivity [1] the Fermi systems with predominant attractive forces possess the property of superfluidity.

The general tendency of predominant attractive forces reveals itself in the conditions of superfluidity of nuclear matter although the conditions of superfluidity for the most general form of nucleon-nucleon interaction are rather complicated and the dependence on the potential is rather complicated too. The existence of the superfluid state of nuclear matter is connected with the behaviour of the even states $J_{st}(k_F)$, $J_{ts}(k_F)$, $J_{sT}(k_F)$, $J_{sl}(k_F)$ and odd states $J_{ss}(k_F)$ $J_{st}(k_F)$, $J_{tt}(k_F)$, $J_{tT}(k_F)$, $J_{tl}(k_F)$ on the Fermi surface. In particular cases when only p-p interactions or central interactions are considered, the conditions of superfluidity are reduced to the requirement that the potentials

$$J_{ts}^{l=0}(k_F), \quad J_{ts}^{l=2}(k_F), \quad J_{ss}^{l=1}(k_F), \quad J_{tt}^{l=1}(k_F), \quad J_{tt}^{l=1}(k_F) + 0.8\,J_{tT}^{l=1}(k_F)$$

are attractive potentials. Since the states with $l > 2$ make a considerably small contribution into nucleon-nucleon interactions of the corresponding energy region no restrictions exist for them.

Let us compare the conditions of superfluidity of nuclear matter with the potentials, obtained from the analysis of the experimental data of nucleon-nucleon scattering. It is known that the particle density of nuclear matter is approximately $2.21 \cdot 10^{38}$ particles/cm^3. The wave length that corresponds to the Fermi energy is then $0.675 \cdot 10^{-13}$ cm. On the basis of a number of

papers (3) the conclusion may be drawn that the potentials have a repulsive character when the wave length is of the order of $(0.2 \div 0.5) \cdot 10^{-13}$ cm and less; at larger wave lengths they have an attractive character. Thus near the Fermi surface the nucleon-nucleon potentials are in general attractive.

According to the data on nucleon-nucleon potentials we may come to the conclusion that the conditions of superfluidity of nuclear matter are in general fulfilled if the interactions leading to superfluidity are weak.

* * *

In conclusion the author considers it a pleasant duty to express his appreciation to N. N. BOGOLJUBOV for his constant interest to the present research and for his valuable remarks.

(3) P. S. SIGNELL and R. E. MARSHAK: *Phys. Rev.*, **106**, 832 (1957); J. L. GAMMEL and R. M. THALER: *Phys. Rev.*, **107**, 291, 1337 (1957).

RIASSUNTO (*)

Si applicano allo studio delle proprietà della materia nucleare i metodi sviluppati nella teoria della superconduttività. Si assume debole l'interazione che conduce alla superfluidità della materia nucleare. L'interazione nucleone-nucleone è presa nella forma più generale. Si trovano soluzioni asintotiche del sistema di equazioni che permettono di ottenere le condizioni di superfluidità della materia nucleare. Le condizioni di superfluidità si riducono in genere alla richiesta che i potenziali nucleone-nucleone al livello dell'energia superficiale di Fermi siano prevalentemente attrattivi. I dati sui potenziali nucleone-nucleone permettono di decidere se le condizioni di superfluidità della materia nucleare sono soddisfatte.

(*) *Traduzione a cura della Redazione.*

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The Scattering Constant for Multiply-Charged Particles in Photographic Emulsion (*).

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(ricevuto il 21 agosto 1958)

Summary. — Numerical values have been calculated for the scattering constant for multiply charged particles in photographic emulsion. The constant has been calculated both with and without cut-off being applied for correction for single scattering, and the results are presented in graphical form.

1. – Introduction.

From measurements made along the track produced by a fast charged particle in nuclear emulsion, the momentum of the particle may be calculated, using the relation

$$p\beta = Kzt^{\frac{1}{2}}/\bar{\alpha}\,,$$

where p is the momentum of the particle having a charge ze, $v = \beta c$ is its velocity and $\bar{\alpha}$ the mean angle of scattering in a cell of length t. Here, K is the « scattering constant », a slowly varying function of z, β, t and the emulsion composition.

The value of K appropriate to a particular track is usually obtained from calculations made by the Bristol group (Gottstein *et al.* [1]) or by Voyvodic and Pickup [2]. These papers contain graphs showing the variation of K with cell size and velocity; these figures are derived from the theories of Wil-

(*) Supported in part by the Office of Scientific Research, Air Research and Development Command under contract AF18(603)-108.

(1) K. Gottstein, M. G. K. Menon, J. H. Mulvey, C. O'Ceallaigh and O. Rochat: *Phil. Mag.*, **42**, 708 (1951).

(2) L. Voyvodic and E. Pickup: *Phys. Rev.*, **85**, 91 (1952).

LIAMS ([3,4]) and MOLIÈRE ([5,6]). However, all of the calculations have been carried out and tabulated for singly charged particles and no calibration experiment has been performed using multiply charged particles.

Following its discovery in 1948, the heavy particle component of the cosmic radiation has been the subject of much research with nuclear emulsions. The charge distribution has been examined, and, sometimes, the energy spectrum, using multiple scattering and ionization measurements. Most attention has been given to the α-particles which are the most plentiful group, and, recently this has led to the finding of irregularities in the expected geomagnetic cut-off in the energy spectrum. It is also of interest to compare the energy spectra of particles of differing charges, and it is thus clearly desirable that energy estimates on these particles should be free from sources of systematic error; the scattering constant used in the scattering measurements appears to be suspect. It has been customary to use a value $K = 32$ for all measurements on particles having $z > 1$, but there appears to be no justification for this in the literature.

It is the purpose of this note to show that, for the scattering cell lengths used, the value of K_{∞} is rather lower. From the information given below, it is possible to calculate the value of K for any particular case.

2. – Calculations.

Based upon the theory of Molière, the scattering constant has been calculated from the expression

$$K_0 = 8.310 B^{\frac{1}{2}}[1 + 0.982/B + 0.117/B^2]$$

as given by GOTTSTEIN *et al.* The constant has been altered to take into account the slightly different emulsion composition. Throughout this note, the appropriate figures have been derived from those given in Table I. These refer to emulsion of density 3.815 g/cm^3, corresponding to a relative humidity of 61%, as is used in the calculations of the standard emulsion range-energy relation (BARKAS ([7])). The parameter B appearing above is a function of the number of collisions suffered by the particle in a cell of length t, and depends also upon z and β.

([3]) E. J. WILLIAMS: *Proc. Roy. Soc.*, A **169**, 531 (1939).
([4]) E. J. WILLIAMS: *Phys. Rev.*, **58**, 292 (1940).
([5]) G. MOLIÈRE: *Zeits. f. Naturfor.*, **2a**, 133 (1947).
([6]) G. MOLIÈRE: *Zeits. f. Naturfor.*, **3a**, 78 (1948).
([7]) W. BARKAS: *Nuovo Cimento*, **8**, 201 (1958).

In performing the calculations, it is necessary to use suitable weighted mean values for terms involving the emulsion composition. So, for example, a term $(NZ^2)^{\frac{1}{2}}$ for a pure substance is replaced by $[\sum_i N_i Z_i^2]^{\frac{1}{2}}$. More complicated averaging is required elsewhere.

An alternate approach is that of VOYVODIC and PICKUP. These authors give an expression for the number of collisions, which is equal to the Williams values for the limiting cases of $\gamma = zZ/137\beta \gg 1$ and $\gamma \ll 1$, and which also holds in between. This expression is derived from one given by GOLDSCHMIDT-CLERMONT (8). Here again, average values must be employed for constants relating to the emulsion composition, and this is adequately dealt with in the paper of VOYVODIC and PICKUP.

TABLE I. – *Composition of emulsion at* $\varrho = 3.815$ g/cm^3.

Element	g/cm^3	$N \cdot (10^{20})$	$NZ \cdot (10^{22})$	$NZ^2 \cdot (10^{24})$
Ag	1.8088	101.01	47.476	22.314
Br	1.3319	100.41	35.143	12.300
I	0.0119	0.565	0.299	0.158
C	0.2757	138.30	8.298	0.499
N	0.0737	31.68	2.218	0.155
O	0.2522	94.97	7.597	0.608
S	0.0072	1.353	0.216	0.035
H	0.0538	321.56	3.216	0.032
	3.815			$36.101 \cdot 10^{24}$

The scattering constant was computed using both of the above approaches, and the results agree to within 1%, with those from the Molière method being systematically larger by this amount. The graphs presented here refer to the results of the Voyvodic and Pickup theory, as these are, in some ways, more accessible. The values of the scattering constant given are reliable to 1%. Values have been computed for protons, α-particles and the limiting case of nuclei with very large charge. It is found that this limiting case is very closely approached for $z = 3$ (to within 1%), and effectively reached for $z = 4$ (within 0.3%). For these high charges, the dependence on β is negligible.

It is desirable to eliminate deflections which are not due to multiple scattering, and it is customary to employ some form of cut-off. There are two forms of cut-off in general use. In the one, all scatters larger than four times

(8) Y. GOLDSCHMIDT-CLERMONT: *Nuovo Cimento*, **7**, 331 (1950).

the mean are completely eliminated; in the other these large scatters are eliminated and replaced by a value equal to four times the mean, *i.e.* one removes only that part of the scatter in excess of four times the mean value. It is possible to calculate the relation between the mean values of the deflection without any cut-off and with the rigorous cut-off; this corresponds to integrating the scattering probability function from zero up to four times the mean value. Explicit calculation of the value of K with cut-off and replacement is not easy, and it is preferable to use an experimental calibration for the difference between the values of K as calculated by these two cut-off methods.

Expressing the deviations in terms of δ, an angular unit common to all theories ($\delta = 2e^2(NZ^2)^{\frac{1}{2}}zt^{\frac{1}{2}}/pv$), VOYVODIC and PICKUP give the following relation between $\overline{\alpha}_0$ and $\overline{\alpha}_{c0}$, where $\overline{\alpha}_0$ is the mean value without cut-off and $\overline{\alpha}_{c0}$ the value with rigorous cut-off.

$$\overline{\alpha}_{c0} = \overline{\alpha}_0 \frac{(1 - (\pi/4\overline{\alpha}_0^2))}{(1 - (\pi/32\overline{\alpha}_0^2))}.$$

This has the result of reducing the value of the scattering constant, K_0, and the resulting values, K_{c0}, are also shown in the figures.

Also included in the values of K_0 and K_{c0} shown, is the contribution due to electron scattering. This has been taken into account by replacing Z^2 by $Z(Z+1)$.

The effect of the finite size of the nuclei is negligible, and there seems no evidence for any saturation of K with increasing cell-size, as has been suggested by some authors.

To eliminate the effects of C-shaped distortion, third differences are often used. These are larger than second differences by slightly more than the commonly used factor of $(3/2)^{\frac{1}{2}}$. ROSENDORFF and EISENBERG [9] have pointed out that this factor should be increased by a few percent, and also give references to experimental calibrations. Our values in this paper refer to second differences.

The change of emulsion composition with the loss or gain of water arising from changes in the external relative humidity can produce noticeable changes in the scattering constant. This will be more serious for emulsions exposed to particle beams from machines under vacuum conditions. Fig. 4 shows the change in scattering constant as a function of the relative humidity; data supplied by SWINNERTON and WALLER [10] of Ilford's has been used for this.

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[9] S. ROSENDORFF and Y. EISENBERG: *Nuovo Cimento*, **7**, 23 (1958).
[10] A. J. SWINNERTON and C. WALLER: Mimeographed report (1957).

3. – Results and discussion.

In Figs. 1, 2 and 3 are displayed the results, calculated as above for second differences, with and without cut-off and including the electron scattering, for protons, α-particles and heavy nuclei. The correction factor which depends on emulsion composition (via relative humidity) is shown in Fig. 4.

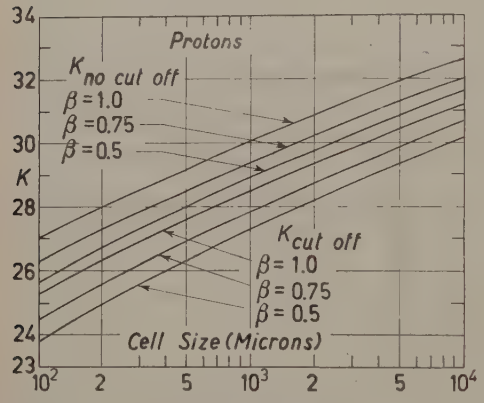

Fig. 1. - Scattering constants K_0 and K_{co} as a function of cell size, for protons having velocities $\beta c = 0.5,\ 0.75,\ 1.0$.

An accuracy of 1% in the values of K is set by the accuracy with which some of the constants of the calculations are known and the degree to which the single scattering theory has been tested. The calculations have all been performed to this final accuracy.

There have been several calibration experiments, of which the results are scattered about the calculated values for K. The agreement has been fair, with some fluctuations, and is has seemed reasonable to use the calculated values for uniformity. Some of these are summarized by VOYVODIC [11]; later measure-

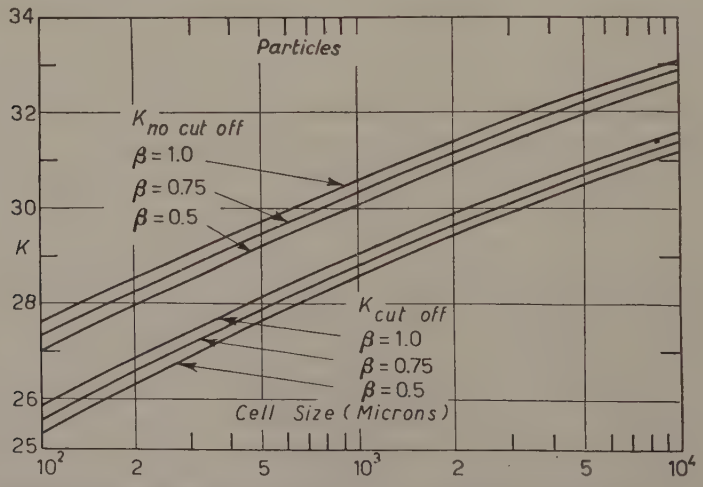

Fig. 2. - Scattering constants K_0 and K_{co} as a function of cell size, for α-particles having velocities $\beta c = 0.5,\ 0.75$ and 1.0.

[11] L. VOYVODIC: *Prog. Cosmic Ray Phys.*, **2**, (1954).

ments were performed by BRISBOUT *et al.* [12] and BISWAS *et al.* [13] at energies of (4÷6) GeV.

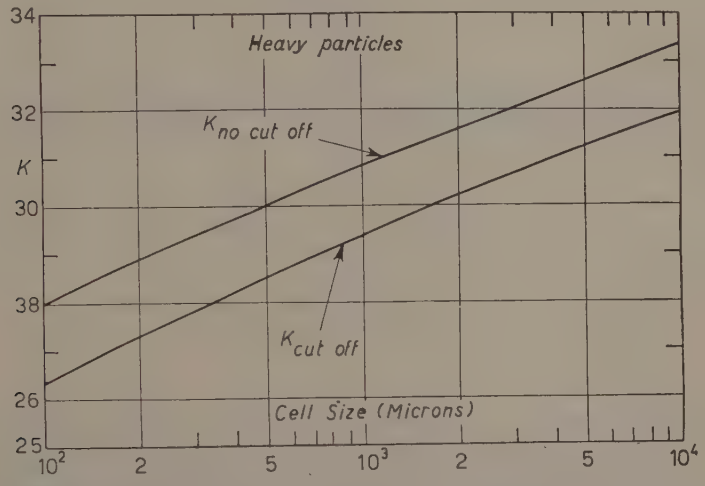

Fig. 3. - Scattering constants K_0 and K_{∞} as a function of cell size, for very heavy nuclei (limiting case). The velocity dependence vanishes for these particles.

An important distinction exists between these two last experiments. The emulsions used by BRISBOUT *et al.* were exposed to the 4.49 GeV/c beam of π^--mesons from the Berkeley Bevatron, and good agreement was obtained between the calculated and experimental values of K. The exposure of emulsions to the 6.2 GeV proton beam was that used by BISWAS *et al.*; these emulsions were exposed to the *internal* beam, and it is possible that there was some loss of water by the emulsions in the vacuum chamber. Certainly the high values of K obtained by BISWAS *et al.* can be accounted for by assuming a more dense emulsion (see Fig. 4). No figures were given by these authors for the emulsion density.

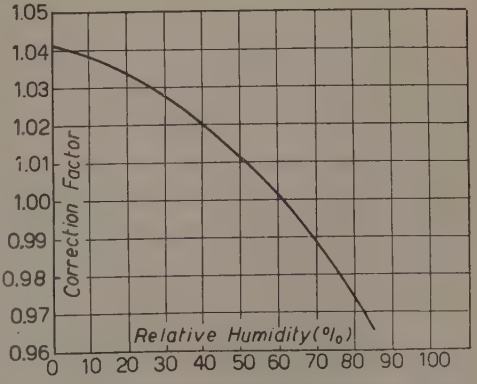

Fig. 4. - Correction factor to be applied, as a function of the relative humidity with which the emulsion is in equilibrium. The scattering constant is to be multiplied by the correction factor.

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[12] F. A. BRISBOUT, C. DAHANAYAKE, A. ENGLER, P. H. FOWLER and P. B. JONES: *Nuovo Cimento*, **6**, 1400 (1956).

[13] S. BISWAS, N. DURGA PRASAD and S. MITRA: *Proc. Ind. Acad. Sci.*, **46**, 167 (1957).

It is not at all clear that the contribution from electron scattering (about 2% increase) and the effect of using a replacement cut-off have been included in the values of K generally used. The difference between K_0 and K_{c0} is calculated to be between 4% and 7%. Some preliminary tests on a small sample of tracks indicates that the scattering constant should be increased by about $(3 \div 4)\%$ to allow for this difference in cut-off; further calibration is in progress. The results of BISWAS *et al.* indicate that the difference between K_0 and K_{c0} is about 7%.

RIASSUNTO (*)

Si sono calcolati i valori numerici della costante di scattering in emulsione fotografica delle particelle con carica multipla. La costante è stata calcolata con e senza la correzione per il cut-off per scattering singolo e i risultati si presentano in grafici.

(*) *Traduzione a cura della Redazione.*

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Diffusion of ^{214}Pb in Nuclear Emulsions.

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(ricevuto il 25 Agosto 1958)

Summary. — A study is made of the diffusion in nuclear emulsions of ^{214}Pb and ^{214}Bi. It is shown that the diffusing nuclide is ^{214}Pb although a small diffusion of ^{214}Bi is not excluded. The order of magnitude of the coefficient of diffusion of ^{214}Pb is found to be 10^{-12} c.g.s. The percentage of non diffusing ^{214}Pb atoms is found to be of the order of 50% in the examined plates.

Introduction.

In this paper a preliminary study of ^{214}Pb diffusion in nuclear emulsion is made.

Many α-particle stars which originate from ^{222}Rn disintegration in nuclear emulsion appear in the form of two branch stars (α-particles from ^{222}Rn and ^{218}Po) plus an isolated track (α-particles from ^{214}Po) (1). This should result from diffusion of one or both of ^{214}Pb and ^{214}Bi nuclides because the ^{214}Po half-life ($15 \cdot 10^{-4}$ s) is too short to allow for its diffusion.

In this paper we analyse these separated stars. Our conclusions can be summarized as follows:

1) The observed separations are due essentially to diffusion of ^{214}Pb although the possibility of a small diffusion of ^{214}Bi is not excluded.

2) About 50% of ^{214}Pb atoms that originate from the ^{218}Pb in nuclear emulsion do not diffuse in our conditions of work.

(1) E. Frota-Pessôa: *An. Ac. Bras. Cie.*, **25**, 337 (1953).

3) The order of magnitude of the ^{214}Pb diffusion coefficient at 0 °C in our conditions of work is 10^{-12} C.G.S.

4) The ^{214}Pb diffusion coefficient increases with increase of temperature as should be expected.

5) The diffusion coefficient seems to decrease with the age of the emulsion used.

1. – Nature of the migrating element.

A G-5 nuclear emulsion was placed in the upper part of an aluminum tube 14 cm high and 3.5 cm in diameter. In the lower part of the tube there was about 600 mg of a pulverized Uranium mineral (250 mg of radium per ton) (2) upon which a lead plate 1 cm thick was placed. This plate, having thin perforation, protected the emulsion from the action of the γ-rays but allowed the radon to go easily to the upper part of the tube. The emulsion face of the nuclear plate was protected by a stainless steel plate in order to lessen the background of α-particles originating from disintegrations in air. The Radon could penetrate the emulsion through a thin air layer between the emulsion and the steel plate. Finally the tube was closed with a rubber plug and protected with paraffin. The same procedure was adopted for the exposure of C-2 plates, to be referred to in Part II.

The exposure of the G-5 plate had a duration of three days at a temperature around 12 °C. After development the 3-branch Radon stars wehre observed as well as the separated ones (1). In order to find out which of the elements ^{214}Pb and ^{214}Bi diffuses, an analysis of the electrons associated with the 3-branch and 2-branch stars and the isolated α near the 2-branch stars was made. The results are given in Table I for 150 3-branch stars and 150 separated ones (2-branch star plus isolated track).

TABLE I.

No. of betas	3-branch	2-branch	isolated
0	45	143	47
1	75	7	71
2	29	0	32
3	1	0	0

(2) F. A. G. A. BRANDÃO, E. FROTA-PESSÔA, N. MARGEM and W. PEREZ: *Proceedings of Geneva Conference*, **8**, 274 (1955).

About 5% of the observed cases were considered doubtful since, due to occasional spots or background betas crossing the origin of the α tracks, it was difficult to establish where the β-rays originated; these were not included in Table I. It should be noticed that in our conditions of work the emulsion did not register minimum ionization electrons, therefore only the low energy part of the spectra of the beta emitters (^{214}Pb and ^{214}Bi) and the low energy conversion electrons were observed.

Table I shows that the distribution of electrons originating from the isolated α-tracks is statistically identical to the distribution of electrons in the 3-branch stars, the 2-branch ones having practically no electrons associated. Thus we should conclude that almost all electron-emitters are associated with the isolated track.

Therefore, if we notice that:

a) The α-disintegrations of ^{222}Rn, ^{218}Po and ^{214}Po are not followed by γ-emission which could produce conversion electrons ([3]);

b) The electron-emitter ^{214}Pb should produce more observable electrons than ^{214}Bi in our conditions of work ([4-6]);

the results of Table I indicate that ^{214}P is the diffusing element, ^{214}Bi having a much smaller diffusion coefficient (their half-lives are of the same order of magnitude). Indeed, if ^{214}Bi had a significant diffusion the electrons from ^{214}Pb would be lost from the isolated α due to ^{214}Po and the distribution of the electrons associated with this track would be different from those corresponding to the 3-branch stars.

Finally we should mention that the electrons associated with the 2-branch stars can be explained as due to the small diffusion of ^{218}Po already mentioned by DEBEAUVAIS *et al.* ([7]). Indeed we have found that about 3% of the separated stars (both for the G-5 and C-2 emulsions) have the longer track due to ^{214}Po in the 2-branch star (*). Only three of those cases could not be explained in this way as they had an electron associated with the 2-branch star and another one with the isolated track.

([3]) F. ASARO and I. PERLMAN: *Rev of Modern Phys.*, **26**, 456 (1954).
([4]) S. KAGEYAMA, *V. of Phys. Soc. of Japan*, **8**, 689 (1953).
([5]) R. A. RICCI and G. TRIVERO: *Nuovo Cimento*, **2**, 745 (1955).
([6]) S. A. E. JOHANSSON: *Arkiv. for Fysik*, **9**, 561 (1955).
([7]) M. DEBEAUVAIS, E. PICCIOTTO and S. WILGAIN: *Nuovo Cimento*, **5**, 260 (1957).
(*) We have also observed occasionally Radon stars with the three branches separated.

2. – Study of the diffusion of ^{214}Pb.

2·1. *Experiment.* – An Ilford C-2 plate 1 in. × 3 in. and 100 μm thick was divided in two parts, each of them was placed in a tube in the same conditions as described in Sect. 1. The emulsion was four months old and had been stocked at 12 °C, 100% humidity.

One of these tubes was placed in a Dewar bottle with a water and ice mixture at 0 °C; the other was placed in a bath, controled by a thermostat at 30 °C. After 3 days exposure in these conditions the plates were developed. 45 days later, this experiment was repeated with another C-2 plate of the same box. In this case, however, the two parts of the plate were weighed, both before and after the exposure, in order to determine the variation in water content. This water content did not change for the plate kept at 0 °C. The plate kept at 30 °C did lose, however, an amount of water corresponding to 0.25 mg per square centimeter of emulsion surface.

2·2. *Observations*

a) As a preliminary study all four plates referred to in Sect. 1 were scanned in order to find the proportion of stars visibly affected by diffusion (2-branch stars + isolated α) to the total number of stars, The visibly separated stars correspond to those cases in which the distance of the 2-branch star vertex from the nearest extremity of the isolated track is larger than 0.5 μm. For each plate a total number of 500 stars were observed using 100 × 12 magnification. The following results were obtained:

First exposure:

(1)

Plate *A* (0 °C)	25%
Plate *B* (30 °C)	47%

Second exposure:

(2)

Plate *a* (0 °C)	27%
Plate *b* (30 °C)	46%

It was also observed that the average value of the separation distances should correspond to a few microns.

These results give already an indication that, in this case, also a significant proportion of ^{214}Pb atoms are retained in the point where they origi-

nate, not being able to diffuse, as has been previously found in Rn (*). The decomposition of the 3-branch stars into those for which ^{214}Pb does not diffuse and those for which there is a small diffusion will result from the analysis given in Sect. 2·4.

b) For each plate 200 visibly separated stars were examined in order to determine the distribution of separation distances.

As the determination of the real distance between the vertex of the two-branch star and the nearest extremity of the α-track (separation distance) involves measurement of depth and of the shrinkage factor we decided to express our results in terms of the horizontal projection of the separation distance. Whenever the vertex of the star was at a distance smaller than 5 μm (in the developed emulsion) from the free surface or from the glass, the event was neglected. This is satisfactory since the maximum observed projection is 8.5 μm. In Table II are shown the results, normalized to a total number of

TABLE II.

1 Scale Div. = 0,5 μm	Number of Cases			
Projection in scale divisions	Plate *A*	Plate *B*	Plate *a*	Plate *b*
0 ÷ 1	7	6	15	2
1 ÷ 2	40	17.5	56.5	24
2 ÷ 3	26.5	13.5	19.5	25
3 ÷ 4	14.5	11.5	6.5	15
4 ÷ 5	5.5	10.5	2	11.5
5 ÷ 6	2	8.5	0	6.5
6 ÷ 7	2.5	2.5	0.5	4
7 ÷ 8	1.5	5.5	0	5
8 ÷ 9	0	4	0	4
9 ÷ 10	0	2.5	0	0.5
10 ÷ 11	0	2.5	0	0.5
11 ÷ 12	0.5	4	0	0
12 ÷ 13	0	2.5	0	1
13 ÷ 14	0	2	0	0.5
14 ÷ 15	0	0.5	0	0.5
15 ÷ 16	0	1	0	0
16 ÷ 17	0	0.5	0	0
> 17	0	0	0	0

(*) In the case of the diffusion of Rn we have shown in reference (1) that only 10% of the Rn formed in the emulsion did not diffuse, thereby originating 4-branch stars. We have called attention to the fact that methods of determination of the Rn content of the emulsion by counting the 4-branch stars assuming that all Rn was retained would

100 cases for each plate. The distances were measured with 100×12 magnification for which 1 division of the scale corresponded to 0.5 μm.

The cases with projections smaller than 1 division correspond to visibly separated stars with real distances larger than 0.5 μm where the separation can be decided by differences in depth.

2·3. *Diffusion coefficient of* ^{214}Pb *in nuclear emulsion.* – We wish to determine, from the results of Table II, the value of the diffusion coefficient of ^{214}Pb. In view of the experimental conditions:

a) Thickness of the emulsion much larger than the average diffusion distance;

b) Exclusion of stars with vertex near the emulsion surfaces;

c) Exposure time much larger than the half-life of ^{214}Pb (as well as of ^{218}Po, ^{214}Bi and ^{214}Po);

we may use the following idealization of our problem:

A large number, N_0, of atoms of a radioactive element (mean life τ), initially concentrated at some point, diffuses in an indefinite medium (diffusion constant D) and stops at the disintegration point (the daughters do not diffuse). What distribution is to be expected for the projected distances between the original position of the diffusing atoms and the daughters, after a time much greater than the lifetime?

TIOMNO has shown (8) that the number of such atoms with projected distances (on a plane) between r and $r+\mathrm{d}r$ is given by

$$\mathrm{d}N = N_0 \lambda^2 r K_0(\lambda r)\,\mathrm{d}r\,, \tag{3}$$

where

$$\lambda^2 = (D\tau)^{-1} \tag{4}$$

and $K_0(x)$ is the K-Bessel function of order zero (9). He has also shown that the number of daughter atoms with projected distances larger than a value

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lead to large errors. In order to overcome this difficulty DEBEAUVAIS, PICCIOTTO and WILGAIN (7) have developed a method based on very low temperature exposures which eliminates the Rn diffusion.

We cannot, however, agree with the statement made by E. PICCIOTTO in the footnote of page 264 of reference (7) that corrections which take into account the loss of Rn were well known as early as 1952. Even now such corrections would not be reliable as the proportion of retained Rn changes very much with the conditions of the emulsion (proportions from 5% to 78% having been quoted in the literature (7)) – E. FROTA-PESSÔA.

(8) J. TIOMNO: private communication.

(9) G. N. WATSON: *A Treatise on Bessel Functions* (Cambridge, 1948), p. 172.

r_0 is given by:

$$N(r_0) = N_0 \lambda r_0 K_1(\lambda r_0) , \tag{5}$$

where $K_1(x)$ is the K-Bessel function of first order ([9]).

Thus we write (3) as

$$\mathrm{d}N/\mathrm{d}r = N(r_0) \frac{\lambda r K_0(\lambda r)}{\lambda r_0 K_1(\lambda r_0)} . \tag{6}$$

Equation (6) has been used to determine the values of λ for ^{214}Pb in the several plates using the results of Table II and taking $r_0 = 1$ scale division.

The advantage of this comes from the fact that we do not need to know the total number of diffusing atoms.

In Fig. 1 the experimental points for $\mathrm{d}N/\mathrm{d}r$ are plotted with the probable errors (statistical), indicated, both for plates A and B, for $N(r_0) = 100$. In

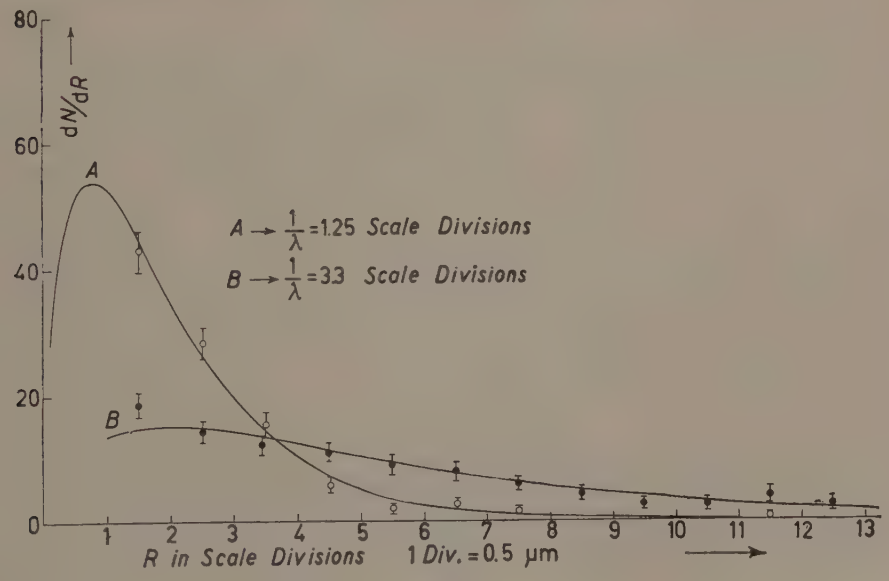

Fig. 1.

Fig. 2, they are plotted for plates a and b. By comparison with a set of theoretical curves (6) for several values of λ we have found the best fits of the experimental points and the corresponding λ values. The continuous curves and the corresponding λ values indicated in Fig. 1 and Fig. 2 correspond to these best fits. From these values and the mean life of ^{214}Pb

$$\tau = 38.7 \text{ min}$$

we obtain the values of the diffusion coefficient of Pb as given in Table III (*) (Equation (4))

TABLE III.

Plate	Temperature	λ^{-1} (scale divisions)	$D \cdot 10^{12}$ (C.G.S.)
A	0 °C	1.25 ± 0.09	1.7 ± 0.24
B	30 °C	3.33 ± 0.25	11.7 ± 1.7
a	0 °C	0.75 ± 0.06	0.6 ± 0.1
b	30 °C	2.0 ± 0.15	4.3 ± 0.6

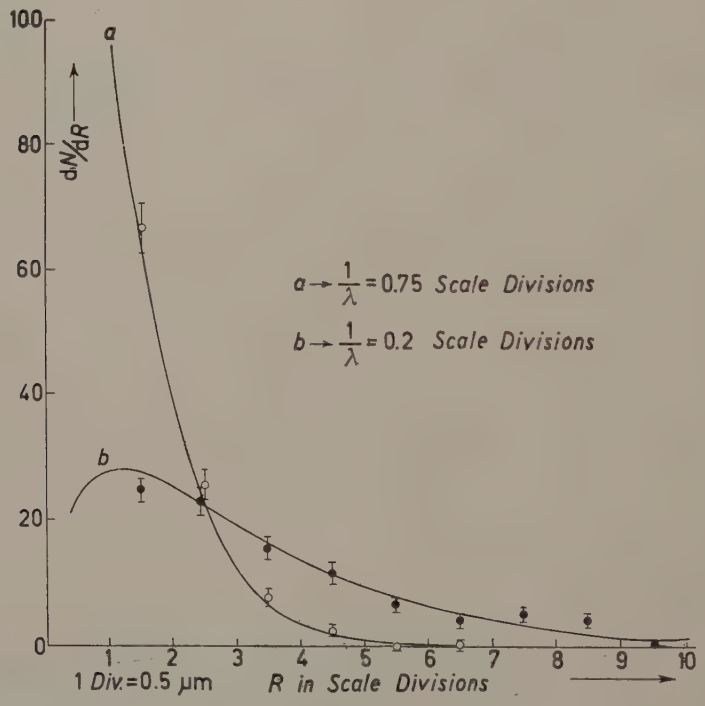

Fig. 2.

We see that D increases with the temperature as should be expected. It also depends on the age of the emulsion, only by a proportionality constant

$$\frac{D_A}{D_a} \simeq \frac{D_B}{D_b} = 2.7 ,$$

(*) The relative error in $1/\lambda$ was taken equal to $1/\sqrt{N}$ where N is the total number of observed stars with separation distances r larger than 1 scale division.

The values of the diffusion coefficient for ^{214}Pb (Table II) should be compared with the value

$$D \simeq 10^{-7} \text{ c.g.s.}$$

found by EICHHOLZ and FLANK ([10]) for Thoron. As should be expected Tn, being an inert gas, has a much larger diffusion coefficient.

2·4. *Proportion of non-diffusing* ^{214}Pb *atoms.* – From equation (5), knowing the number $N(r_0)$ of atoms which diffused more than r_0 (in projection) and the value of λ we obtain the total number N_0 of diffusing Pb atoms ($r_0 = 1$ scale division). From results (1), (2) we obtain the numbers N_3 of visually non-separated 3-prong stars (retained Pb atoms + non distinguishable separated stars) corresponding to $N(r_0) = N(1)$. Indeed results (1) and (2) correspond to

$$\frac{N(1) + N_{0-1}}{N(1) + N_{0-1} + N_3},$$

where N_{0-1} are the observed separated stars in the range 0–1. Adding N_{0-1} to N_3 and subtracting $N_0 - N(1)$ we find the number of retained Pb atoms (which were not able to diffuse):

$$N_{\text{Ret}} = N_3 + N_{0-1} + N(1) - N_0 .$$

The results are given in Table IV where the proportions k of retained Pb atoms to the total number of cases are also given,

$$k = \frac{N_{\text{Ret}}}{N_0 + N_{\text{Ret}}} .$$

TABLE IV.

Plate	N_{0-1}	$N(1) = N(r_0)$ $r_0 = 1$ s.d.	N_0	$\frac{N(1) + N_{0-1}}{N(1) + N_{0-1} + N_3}$	N_3	N_{Ret}	k
A	7	93	135	0.25	300	265	0.66 ± 0.022
B	6	94	102	0.47	113	111	0.52 ± 0.022
a	15	85	181	0.27	270	189	0.51 ± 0.04
b	2	98	118	0.46	117	99	0.46 ± 0.026

We see that the proportion of retained ^{214}Pb atoms is of the order of 50% in our conditions of work.

([10]) G. G. EICHHOLZ and F. C. FLANCK: *Journ. Phys. Chem.*, **19**, 363 (1951).

The last column of Table IV gives the fractions k of retained ^{214}Pb atoms with probable errors. These results indicate that k increases when the temperature decreases and seem to indicate that k decreases for older plates.

* * *

We are thankful to Professors J. TIOMNO and U. CAMERINI for helpful discussions, to DELIA V. FERREIRA for help in measurements and numerical computations and to Miss CLOTYLDE Z. BILBAO for help in measurements.

RIASSUNTO (*)

Si è studiata la diffusione del ^{214}Pb e del ^{214}Bi in emulsioni nucleari. Si dimostra che il nuclide che si diffonde è il ^{214}Pb, benchè non si escluda una debole diffusione del ^{214}Bi. L'ordine di grandezza del coefficiente di diffusione del ^{214}Pb risulta 10^{-12} c.g.s. Si è trovato che la percentuale di atomi di ^{214}Pb non diffusi è, nelle lastre esaminate, dell'ordine del 50%.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

A Possible Method for the Determination of the Parity of Strange Particles.

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(ricevuto il 28 Agosto 1958)

Summary. — A method for the determination of the parities of K-mesons and hyperons based on the measurement of left-right asymmetry in reactions $\pi+p$, $Y+K$ on a polarized proton target is proposed.

The weak interactions which are responsible for the decay of the strange particles do not conserve parity, hence the intrinsic parity of these particles can be determined only by means of an analysis of their strong interactions.

We propose a method for determining the intrinsic parity of K-mesons and hyperons, based on the study of the reactions:

$$(1) \qquad \begin{cases} \pi + p \to \Lambda(\Sigma) + K \\ K + p \to \Lambda(\Sigma) + \pi \end{cases}$$

on a *polarized* proton target. In conformity with present day data we consider the hyperon spin as $\frac{1}{2}$ and the K-meson spin as 0. The transition matrix for processes of the type (1) has the general form:

$$(2) \qquad M = a + \boldsymbol{b}\boldsymbol{\sigma} .$$

The density matrix of the initial state is

$$(3) \qquad \varrho_0 = \tfrac{1}{2}(1 + \boldsymbol{P}_0\boldsymbol{\sigma}) ,$$

where $\boldsymbol{P}_0$ is the polarization of the proton target.

From (2) and (3) one can deduce the following expression for the differential cross-section

$$\sigma(\theta, \varphi) = (aa^* + \boldsymbol{b}\boldsymbol{b}^*)\left(1 + \boldsymbol{P}_0 \frac{a^*\boldsymbol{b} + a\boldsymbol{b}^* + i\boldsymbol{b}^* \times \boldsymbol{b}}{aa^* + \boldsymbol{b}\boldsymbol{b}^*}\right). \tag{4}$$

For an unpolarized target the following polarization of the hyperons results:

$$\boldsymbol{P} = \frac{a\boldsymbol{b}^* + a^*\boldsymbol{b} + i\boldsymbol{b} \times \boldsymbol{b}^*}{aa^* + \boldsymbol{b}\boldsymbol{b}^*}. \tag{5}$$

According to the intrinsic parity of the reacting particles one can distinguish two possibilities:

1) The intrinsic parity does not change, *i.e.*

$$I_\pi I_p = I_Y I_K .$$

In this case the matrix (2) is a scalar and

$$\boldsymbol{b} = b_0(\boldsymbol{k} \times \boldsymbol{k}') , \tag{6}$$

where $\boldsymbol{k}$, $\boldsymbol{k}'$ are unit vectors in the directions of the relative momenta of the initial and final states respectively. From (4), (5) and (6) we obtain

$$\sigma = \sigma_0(1 + \boldsymbol{P}_0\boldsymbol{P}); \qquad \boldsymbol{P} = \frac{a\boldsymbol{b}_0^* + a^*\boldsymbol{b}_0}{aa^* + \boldsymbol{b}\boldsymbol{b}^*}(\boldsymbol{k} \times \boldsymbol{k}') , \tag{7}$$

where σ_0 is the cross-section for an unpolarized target.

2) The intrinsic parity changes, *i.e.*

$$I_\pi I_p = -I_Y I_K .$$

The matrix (2) is now pseudoscalar and

$$a = 0 ; \qquad \boldsymbol{b} = b_1\boldsymbol{k} + b_2\boldsymbol{k}' \tag{8}$$

in this case

$$\sigma = \sigma_0(1 - \boldsymbol{P}_0\boldsymbol{P}); \qquad \boldsymbol{P} = i\frac{b_1b_2^* - b_2b_1^*}{\boldsymbol{b}\boldsymbol{b}^*}(\boldsymbol{k} \times \boldsymbol{k}') . \tag{9}$$

Choosing the y-axis along the direction of the vector of initial polarization $\boldsymbol{P}_0$ one can write eq. (9) in the following final form

$$\sigma(\theta, \varphi) = \sigma_0(1 \pm P_0P \cos\varphi) . \tag{10}$$

This yields the following value for the asymmetry

$$e(\theta) = \frac{\sigma(\theta, 0) - \sigma(\theta, \pi)}{\sigma(\theta, 0) + \sigma(\theta, \pi)} = \pm P_0 P \,. \tag{11}$$

This is the main result of this note. I t is obvious from eq. (11) that the sign of the asymmetry determines the product of the intrinsic parities $I_Y I_K I_p$: In order to determine this product, one must measure the sign of the hyperon polarization P resulting from reactions (1) on an unpolarized target (the initial polarization P_0 is known).

It is natural to use the decay of the polarized hyperon as an analyser [1]. The asymmetry of the π-mesons resulting from the decay determines the product αP (α is the asymmetry coefficient [2]). α can be determined by measuring the *polarization of the protons* resulting from the hyperon decay [1] (*).

Thus, using the sign of the polarization of hyperons from a reaction (1) with an unpolarized proton target, determined from the subsequent decay of the hyperons, we can determine uniquely the product of intrinsic parities $I_Y I_K I_p$ from experiments on a polarized target. We emphasize the fact that the main result (Eq. (11)) is rigorous and is based only on the following assumptions: 1) parity conservation in strong interactions in which strange particles participate; 2) the spin of the hyperons is $\frac{1}{2}$, and that of the K-meson 0.

Note that the study of various reactions (1) on polarized proton targets can also yield the relative parities of the Λ and Σ particles. Thus, the reactions

$$\pi^- + p \to \Lambda^0 + K^0$$

$$\pi^+ + p \to \Sigma^+ + K^+$$

yield the values of $I_\Lambda I_p I_K$ and $I_\Sigma I_p I_K$ respectively, from which $I_\Sigma I_\Lambda$ can be determined.

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[1] T. D. Lee, and C. N. Yang: *Phys. Rev.*, **108**, 1102 (1957).

[2] T. D. Lee, J. Steinberger, G. Feinberg, P. R. Kabir and C. N. Yang: *Phys. Rev.*, **106**, 1367 (1957); F. S. Crawford, Jr. M. Cresti, M. L. Gottstein, E. M. Lyman, F. T. Solmitz, M. L. Stevenson and H. K. Ticho: *Phys. Rev.*, **108**, 1102 (1957); R. Plano, A. Prodell N. Samios, M. Schwarz, J. Steinberger, P. Bassi, V. Borelli, G. Puppi, G. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, P. Franelli, R. Santagelo, V. Siverstini, D. A. Glaser, G. A. Graves and M. L. Perl: *Phys. Rev.*, **108**, 1340 (1957).

(*) At the International Conference on High Energy Physics (Geneva 1958) the following value for the asymmetry coefficient in the decay of Λ^0-particles, has been reported,

$$\alpha = 0{,}85^{+0{,}15}_{-0{,}21} \,.$$

All the results obtained apply, *mutatis mutandis*, to reactions on polarized nuclei of spin $\frac{1}{2}$, in which hypernuclei of spin $\frac{1}{2}$ are generated.

In conclusion, we remark, that if the intrinsic parities of strange particles are known, the study of the asymmetry in reactions (1) on polarized protons allows the determination of the polarization (magnitude and sign) of the hyperons from an unpolarized target.

* * *

The authors expresses their gratitude to A. A. Logunov for stimulating discussions and his interest in this work.

RIASSUNTO (*)

Si propone un metodo per la determinazione delle parità dei mesoni K e degli iperoni basato sulla misura dell'asimmetria destra-sinistra nelle reazioni $\pi - p$, $Y + K$ su un bersaglio di protoni polarizzati.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

Ultrasonic Absorption in Some Homologous Series of Organic Liquids.

Part III. – Alcohols.

S. Parthasarathy, M. Pancholy and A. F. Chhapgar

National Physical Laboratory of India - New Delhi

(ricevuto il 16 Settembre 1958)

Summary. — Values of ultrasonic velocity and absorption coefficient in a number of alcohols are reported. The relationship between molecular weight on the one hand and absorption coefficient and velocity together with their temperature coefficients on the other, is discussed. It is suggested that the excess absorption in alcohols is mainly due to bulk viscosity. Some general conclusions regarding absorption in homologous series are also given.

In the previous two papers ([1,2]), the relation between absorption and molecular constitution in the series acids, esters and hydrocarbons was discussed. It was seen that in the acid series and the aromatic hydrocarbon series, the absorption decreased with molecular weight whereas in the aliphatic hydrocarbon series the absorption increased with molecular weight. The excess ratio however always decreases with molecular weight in all the series. The present paper deals with the alcohol series.

The measurements were carried out by the pulse method at a frequency of 21 MHz over a temperature range of 40°. The results are shown in Table I.

([1]) S. Parthasarathy, M. Pancholy and A. F. Chhapgar: *Nuovo Cimento*, **10**, 111 (1958).

([2]) S. Parthasarathy, M. Pancholy and A. F. Chhapgar: *Nuovo Cimento*, **10**, 118 (1958).

TABLE I.

Liquid	Velocity a 20 °C m/s	Temp. coeff. of vel.	Viscous absorption · 10^{17} at 20 °C	Observed absorption · 10^{17}				$(1/\alpha) \cdot (d\alpha/dT)$
				20°	30°	40°	50 °C	
Methyl alcohol	1134	— 3.1	13.6	33.7	32.1	30.6	29.1	— 0.0049
Ethyl alcohol	1184	— 3.3	23.8	55.1	51.0	47.2	43.7	— 0.0077
n-Propyl alcohol	1226	— 3.7	37.4	79.3	70.7	63.0	56.1	— 0.012
n-Butyl alcohol	1262	— 3.5	48.5	85.7	77.0	69.1	62.1	— 0.011
n-Amyl alcohol	1291	— 3.5	60.9	109.9	94.2	82.0	69.8	— 0.015
n-Hexyl alcohol	1317	— 3.5	71.8	122.3	104.0	88.3	75.0	— 0.016
n-Heptyl alcohol	1340	— 3.5	90.0	135.2	115.0	98.0	83.3	- 0.016
n-Octyl alcohol	1362	- 3.3	111.4	183.3	144.0	113.0	88.8	— 0.024
n-Decyl alcohol	1398	— 3.5	183.8	230.4	186.0	150.2	121.3	— 0.022

Fig. 1 shows the variation of velocity with temperature. Fig. 2 shows the relation between velocity at 20 °C and molecular weight. This relationship

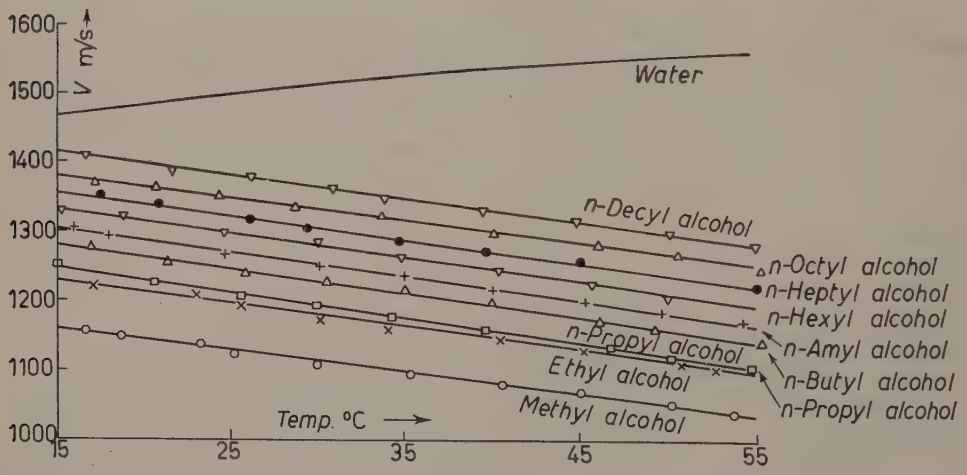

Fig. 1.

is similar to those described earlier (2). The temperature coefficient of velocity also shows a definite relation to molecular weight, the exceptions for methyl and ethyl alcohols being probably due to traces of water vapour.

Fig. 3 shows the variation of absorption with temperature. The absorption is very low and has a very slight negative temperature coefficient for the first member. But the absorption increases rapidly for the higher members and the temperature coefficient also becomes more negative. This is shown more clearly in Fig. 4, where it is seen that the difference between observed absorp-

tion and the calculated absorption remains more or less constant for all liquids. The ratio of the two however decreases with molecular weight. The tempe-

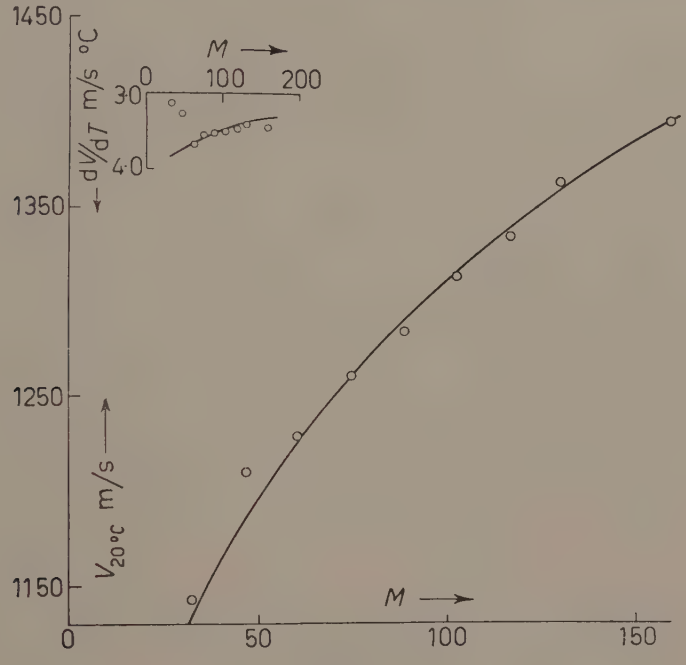

Fig. 2.

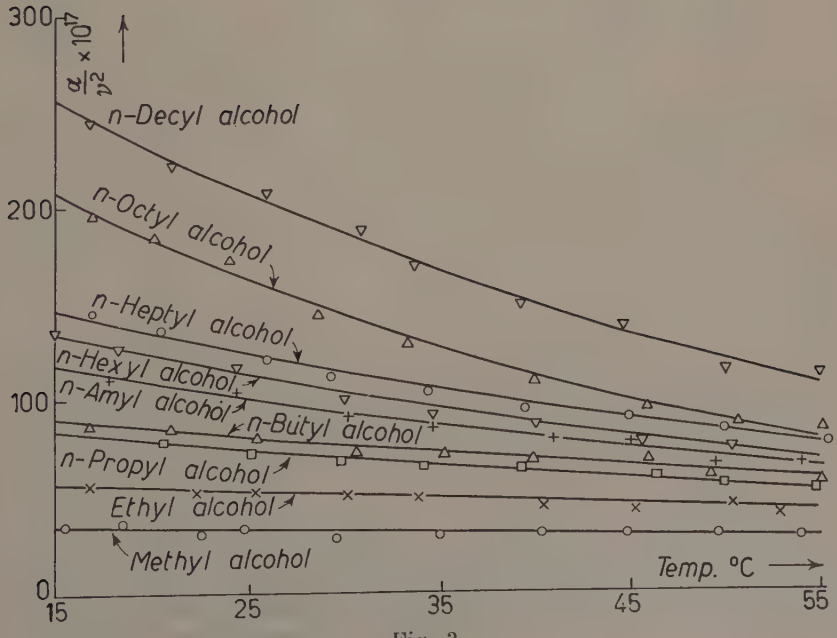

Fig. 3.

rature coefficient of measured absorption is about the same as that due to viscous effects, as seen in Table II. The values of absorption in a few alcohols

TABLE II.

Liquid	Temperature coefficient of absorption	
	Observed $\cdot 10^{17}$	Calculated $\cdot 10^{17}$
Methyl alcohol	— 0.005	— 0.004
Ethyl alcohol	— 0.008	— 0.008
n-Propyl alcohol	— 0.012	— 0.012
n-Butyl alcohol	— 0.015	— 0.011
n-Amyl- alcohol	— 0.010	— 0.015

at one temperature have been given by WILLARD [3], RAPUANO [4], PELLAM and GALT [5], while VERMA [6] has measured the variation of absorption with

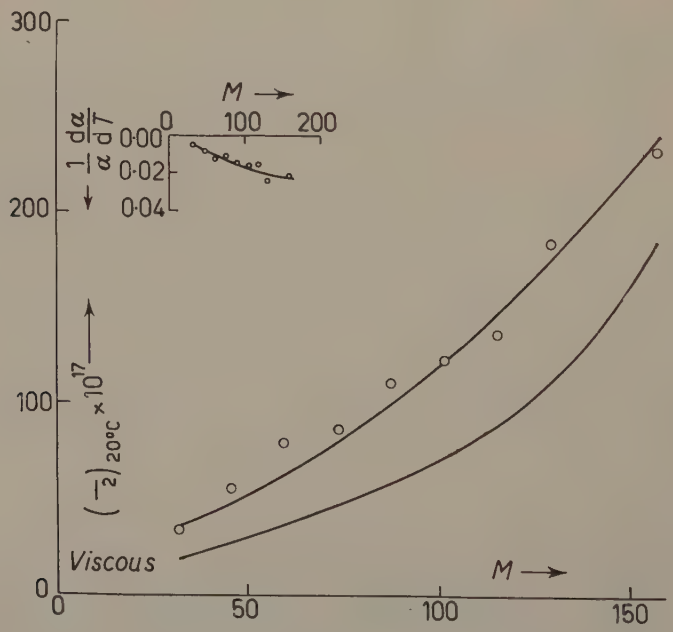

Fig. 4.

[3] G. W. WILLARD: *Journ. Acoust. Soc. Amer.*, **12**, 438 (1941).
[4] R. A. RAPUANO: *Phys. Rev.*, **72**, 78 (1947).
[5] J. R. PELLAM and J. K. GALT: *Journ. Chem. Phys.*, **14**, 608 (1946).
[6] G. S. VERMA: *Journ. Chem. Phys.*, **18**, 1352 (1950).

temperature in a few alcohols and the data given in the present work agree with those of earlier workers.

Data on the variation of viscosity with temperature are not available for the higher members of the series and it is therefore not possible to calculate how the viscous absorption changes with temperature. The data available for the lower members indicate that both excess absorption and excess ratio do not change much with temperature, having only a small negative coefficient.

All these facts show that the behaviour of the alcohols is more or less similar to that of the aliphatic hydrocarbons and the mechanism of absorption in alcohols is probably mainly due to a bulk viscosity effect. In the lower members of the series, the excess absorption is of the same order of magnitude as the viscous absorption. For the higher members of the series, however, the excess absorption becomes very much less than the viscous absorption. This would show that whereas for the lower members both viscosities play an equal part, for the higher members the effect of bulk viscosity diminishes.

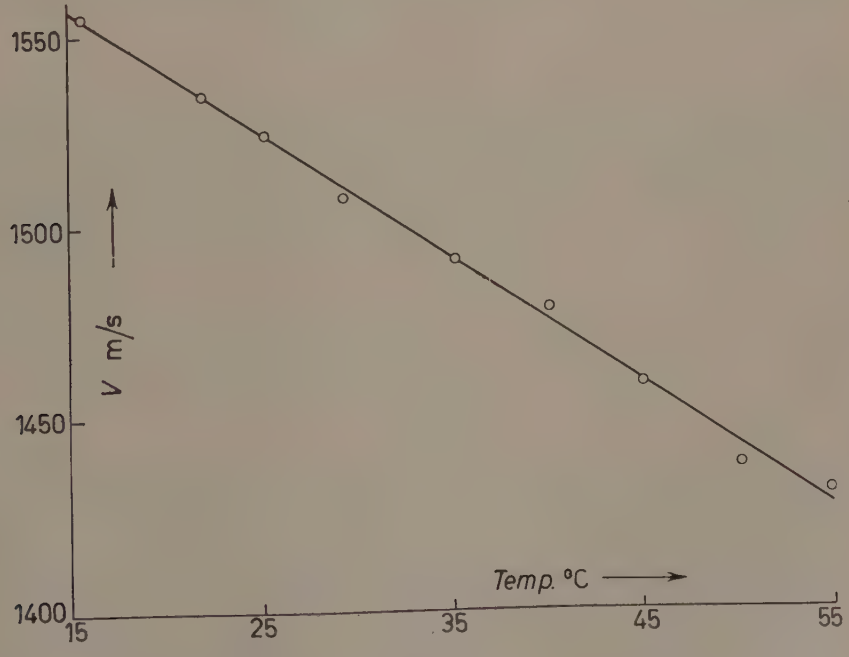

Fig. 5.

In order to compare the behaviour of aromatic alcohols with that of the aliphatic alcohols, measurements were made on benzyl alcohol, the only liquid available in this series, and the results are given in Table III. No previous results on this liquid are known. Figs. 5 and 6 show the temperature varia-

TABLE III.

Liquid	Velocity at 20 °C m/s	Temp. coeff. of vel.	Viscous absorption $\cdot 10^{17}$	Observed absorption $\cdot 10^{17}$				$(1/\alpha) \cdot (d\alpha/dT)$
				20°	30°	40°	50 °C	
Benzyl Alcohol	1541	— 3.2	38.4	96.0	82.5	71.0	61.0	— 0.015

tion of velocity and absorption respectively. These results show that the aromatic alcohol also behaves like the aliphatic alcohols and the cause of absorption is also probably the same. This is in contrast to the behaviour of the parent compounds which have different causes of absorption and behave in a different manner.

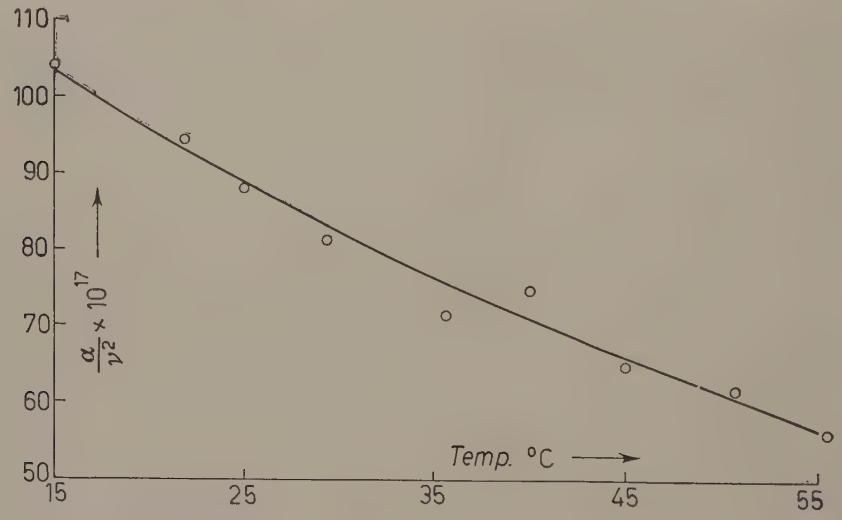

Fig. 6.

The following conclusions emerge from the study of absorption in a homologous series of liquids:

1) Absorption and its temperature coefficient are related to molecular constitution.

2) The absorption increases progressively with molecular weight for liquids showing a viscosity or structural relaxation mechanism, like alcohols and aliphatic hydrocarbons.

3) The absorption decreases progressively with molecular weight for liquids showing a relaxation mechanism, like fatty acids and aromatic hydrocarbons.

4) Where relaxation effects are established, as in fatty acids, although absorption varies with frequency the relation between absorption and molecular weight shows the same trend as above.

5) The excess absorption behaves in the same way as the total absorption for both kinds of mechanisms.

6) The excess ratio always decreases with molecular weight for any series.

7) The relaxation frequencies also have a relation to molecular weight.

8) Absorption in an aromatic series is higher than in the corresponding aliphatic series.

RIASSUNTO (*)

Si riferiscono i valori della velocità degli ultrasuoni e del coefficiente di assorbimento in una serie di alcoli. Si discute la relazione fra il peso molecolare da un lato e il coefficiente di assorbimento e la velocità assieme ai loro coefficienti di temperatura dall'altro. Si presume che l'eccesso di assorbimento negli alcoli sia principalmente dovuto alla viscosità globale. Si espongono anche alcune conclusioni d'indole generale riguardanti l'assorbimento in serie omologhe.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

On the Classification of the New Particles.

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(ricevuto il 29 Settembre 1958)

Summary. — The types of new quantum numbers for elementary particles which result from extending L_{inh} (inhomogeneous Lorentz group) to a larger symmetry group which is not simply $L_{\text{inh}} \times$ isotopic rotations are investigated. The main results are *a*) there is another spin having nothing to do with Lorentz transformations, of the type of isotopic spin; *b*) each irreducible representation of the group comprises a finite family of conventional particles with a certain spectrum of spins; the spectrum of the z-component of this « ultra-Lorentz » spin is related to the spin spectrum in a definite way; *c*) a structure like the empirical division of all particle families into the class of those which coincide with their anti-particle families and the class of those whose antiparticles form different families is found; *d*) a revision of the concept of bare particle mass is implied.

1. – General remarks.

Elementary particles are at present described by fields satisfying irreducible wave equations of the fundamental physical symmetry group. That is, the components of their fields are geometric quantities (scalars, vectors, spinors, etc.) belonging to various irreducible representations or *reps* as we shall say for short, of the symmetry group. Apart from gauge transformations and the isolated improper transformations C, $\mathcal{P}$ and $\mathcal{T}$ this group is taken to be simply the direct product of the inhomogeneous Lorentz group L_{inh} and the group of rotations R_3 in another, 3-dimensional space called isotopic space. For example, the K-mesons K^+, K^0 are assumed to be (probably) ordinary scalars and to form the two components of an isotopic spinor.

The group L_{inh} can attach only the quantum numbers spin and bare par-

ticle mass to particles. The bare particle mass spectrum so obtained is the continuous one ranging from 0 to $+\infty$ and is thus no help whatever in solving the puzzle of the observed discrete mass spectrum. Moreover, as an observationally meaningless concept, bare particle mass will probably have to be purged from a future correct theory. Thus the group L_{inh} had to be widened to $L_{\text{inh}} \times R_3$ to account for the new quantum numbers of isotopic spin. Some of us believe that this manner of extending L_{inh} is only a temporary expedient, that eventually it will be extended in a way which better synthesizes isotopic space and space-time. This better synthesis will illuminate the connection of the two spaces, that is, the physical meaning of isotopic transformations and their possible connection with space-time transformations.

This paper sketches the sort of quantum numbers, *i.e.*, the classification scheme, one gets by extending L_{inh} to a larger symmetry group which is not simply the direct product of L_{inh} and another group, as is the conventional $L_{\text{inh}} \times R_3$. The group G in question is the conformal group on space-time; various of its physical aspects have been explored in previous papers [1]. Although we believe that this particular group has real physical content, this paper may be read as the treatment of a model which illuminates in a fairly typical way how new quantum numbers are generated alongside the old ones when L_{inh} is imbedded in a *simple* larger group.

The group G may be treated as the rotation group on a 6-dimensional flat space of signature $(++---+-)$. Its generator algebra thus has the familiar form of rotation groups. The general procedure is to look among these 15 generators and their various algebraic combinations for the physical quantities of interest. These latter are quantities like spin, isospin, heavy particle number, charge, etc. Their eigenvalues may be called the « true » quantum numbers of a particle; they are half integers, like spin, or ± 1, 0 like heavy particle number, charge in units of e, etc. They do *not* comprise quantities like the mass. On selects from these operators a convenient and physically interesting complete set of commuting observables and simultaneously diagonalizes them, which « reduces out » the various reps with respect to this set. One then finds each rep split into a family of conventional particles, each labelled by a set of these quantum numbers. More generally, the spectrum of any observable of interest in the algebra for a particular rep can be obtained by substituting the matrices of that rep into the observable and then diagonalizing it.

The modest scope of investigating only the true quantum numbers carries certain advantages. These numbers, by their very nature, cannot change when an interaction with other fields is « turned on » or « off », and may thus be looked for in that physical fiction, the bare particle. We bypass the whole

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[1] R. Ingraham: *Nuovo Cimento*, **12**, 825 (1954); *Phys. Rev.*, **101**, 1411 (1956); **106**, 595 (1957); R. Ingraham and J. Ford: *Phys. Rev.*, **106**, 1324 (1957).

difficult and at present unknown domain of the dynamics of the interacting fields by confining attention to the true quantum numbers. The deep question of mass in particular is probably inseparably bound up with dynamics [2].

2. – Details.

It proves very convenient (see just below) to use the fundamental form

$$G_{\mu\nu}X^{\mu}X^{\nu} \equiv X^2 - (X^4)^2 - 2X^0X^5 \tag{1}$$

in which the « ultra-Lorentz » co-ordinates X^0, X^5 have non-principal axes form [3]. Then the 15 generators $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0, 1, ..., 5)$ generate the following infinitesimal transformations:

M_{mn} $(m, n = 1, ..., 4)$	space-time rotations (proper Lorentz transformations)
M_{5m}	space-time translations
M_{0m}	space-time accelerations
M_{50}	uniform space-time dilatations

One can build two commuting angular momentum operators $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ from the 6 M_{mn} in exactly the same way that this is done in extracting the reps of the homogeneous Lorentz group [1]. These two angular momenta also commute with

$$\Delta \equiv \frac{1}{i} M_{50} ,$$

The five operators $\boldsymbol{j}_1^2$, $(j_1)_z$, $\boldsymbol{j}_2^2$, $(j_2)_z$, Δ form a convenient complete set for analysing the reps of G, whose matrix elements are consequently labelled with the set $(j_1, j_2, m_1, m_2, \delta)$, where $j_i(j_i+1)$ and m_i are eigenvalues of $\boldsymbol{j}_i^2$ and $(j_i)_z$ $(i = 1, 2)$ and δ is an eigenvalue of Δ. The spin operator $\boldsymbol{s} = (M_{12}, M_{23}, M_{31})$ turns out to be the vector sum of $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$:

$$\boldsymbol{s} = \boldsymbol{j}_1 + \boldsymbol{j}_2 . \tag{2}$$

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(2) For a clear statement of this position, see W. HEISENBERG: *Rev. Mod. Phys.*, **29**, 269 (1957).

(3) For the meaning of the X^{μ} in terms of space-time, see any of the works cited in footnote (1).

(4) See the works cited in reference (5).

a) When the commutator algebra is explicitly solved for the matrices representing the $M_{\mu\nu}$ by the method of L. H. THOMAS ([5]) or Y. MURAI ([6]), certain algebraic relations between j_1, j_2 and δ result as « by-products ». Some of these come from equations of the generator algebra *i.e.*, they express the fact that the matrices form a representation of the $M_{\mu\nu}$. Others come from further equations expressing that the representation be irreducible, *i.e.*, a rep. A pair of values (j_1, j_2) can therefore occur in a rep associated only with the one or several values of δ which satisfy the above equations. Since Δ is the « z-component » of an angular momentum, δ can have only values of the form $n/2$, n an integer. Because j_1 and j_2 also are (non-negative) half integers, these equations restrict the possible number of triples (j_1, j_2, δ) which can occur in any rep. [These restrictions are the analogues of the conditions $j = \text{const} = \text{half integer}$ for the reps of the rotation group; or $j_1 = \text{const}$, half integer, $j_2 = \text{const}$, half-integer for the reps of L_{hom}.] With this commuting set one therefore gets finite dimensional reps of G.

For every pair (j_1, j_2) occurring in a rep, one gets all the values $s = j_1 + j_2$, $j_1 + j_2 - 1, \ldots, |j_1 - j_2|$ in virtue of (2). All these s are therefore associated with the particular δ which belongs with this (j_1, j_2) in the triple (j_1, j_2, δ). By a change of basis one could label the rows and columns of the reps instead with the 5-ple (j_1, j_2, s, m, δ) where $m =$ eigenvalue of s_z and s takes on all the values given by the vector sum of j_1 and j_2. Thus the reps of G comprise families of conventional particles labelled by their spins ([7]) and by another spin z-component quantum number δ (as well as by two other angular momentum total quantum numbers.)

To illustrate this structure, consider a particularly simple sequence of reps of G which we shall call the α-sequence. The algebraic conditions on j_1, j_2 and δ are simply

$$\left\{\begin{array}{ll} j_1 + j_2 = p & (= \text{non-negative half integer}). \\ j_1 - j_2 = \delta & \end{array}\right. \tag{3}$$

Hence the dimensions of these reps are

$$\mathrm{d}(p) = \sum_{j_1 = 0, \frac{1}{2}, 1, \ldots}^{p} (2j_1 + 1)(2p - 2j_1 + 1) = \frac{(p+1)(2p+1)(2p+3)}{3} .$$

The value $p = 0$ gives the trivial identity rep. For $p = \frac{1}{2}$ the triples $(j_1, j_2, \delta) =$

([5]) L. H. THOMAS, *Ann. Math.* **42**, 113 (1941).

([6]) Y. MURAI: *Prog. Theor. Phys.*, **9**, 147 (1953).

([7]) Cfr. the reps of $L_{\text{inh}} \times R_3$, which describe only single conventional particles. That is because $\mathbf{s}^2$ is in the *center* of the $L_{\text{inh}} \times R_3$ algebra (commutes with all its generators) but not in the center of the G-algebra.

$= (\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{1}{2}, -\frac{1}{2})$ occur in the rep (8); its dimension is 4. Thus this rep contains one particle of spin $\frac{1}{2}$ labelled by $\delta = +\frac{1}{2}$ and one particle of spin $\frac{1}{2}$ labelled by $\delta = -\frac{1}{2}$. For $p = 1$, the dimension $= 10$ and the triples $(j_1, j_2, \delta) = (1, 0, 1)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(0, 1, -1)$ occur. Therefore this rep (8) contains a particle of spin 1 labelled by $\delta = +1$, a particle of spin 1 and one of spin 0 both labelled by $\delta = 0$, and a particle of spin 1 labelled by $\delta = -1$. And so on. From the condition (3) one can see that the α-sequence contains only fermion families and boson families, no mixed families.

The rotation group G itself is a rep outside the α-sequence. This rep is 6-dimensional and contains one particle of spin 1 and $\delta = 0$ and three particles of spin 0 with $\delta = +1, 0, -1$ respectively. This scheme is seen not to conform to (3). Incidentally, the problem of finding *all* the reps of G has never in our opinion been successfully carried through (9).

b) The reps as a whole are labelled by three numbers (q, r, w), the eigenvalues of the three operators Q, R, W comprising the center of the algebra (the set of elements commuting with all the generators $M_{\mu\nu}$). These are respectively of second (10), third, and fourth algebraic degree in the $M_{\mu\nu}$. We give R explicitly because of its importance for this discussion:

$$R \equiv 8 M_{[12} M_{34} M_{50]} = -\frac{8}{6!} \varepsilon^{\mu\nu\lambda\xi\varrho\tau} M_{\mu\nu} M_{\lambda\xi} M_{\varrho\tau} . \tag{4}$$

All the reps of the α-sequence are « doublets » in the sense that they occur in pairs with the same q and w and the same spectrum (j_1, j_2) within the rep; they differ only in that $r_1 = -r_2 \neq 0$. Thus there are two 4-dimensional reps of the kind described explicitly in part *a*, two 10-dimensional reps, and so on. On the other hand, $r = 0$ for the 6-dimensional rep, not in the α-sequence, mentioned in part *a*. It is thus a « singlet ». One can prove that there are only singlets and doublets among the reps of G, *i.e.*, the eigenvalues of R are either zeroes or occur in pairs $\pm r \neq 0$.

This property is related to the following fact. An improper transformation of G changes the sign of r for a rep and thus interchanges the two members of a doublet. This is evident from (4), for R is a 6-dimensional pseudoscalar operator, whereas Q and W are both scalar (invariant) operators. The possible physical meaning of this will be remarked later.

c) Bare particle mass. These particles can *not* be associated with any value of the bare particle mass. This strange feature comes from the fact

(8) Equations (3) give « one component » of the α-sequence. The other component has $j_1 - j_2 = -\delta$. Cfr. part *b*.

(9) It was attempted in reference (6).

(10) Q is the familiar Casimir operator.

that the momentum operators $p_m = M_{5m}$ (generators of infinitesimal translations) are not diagonalizable matrices for any of the reps we have found. Hence $M^2 = (p_4)^2 - \boldsymbol{p}^2$ need not necessarily be diagonalizable. To cite a few cases, $M^2 = 0$ for the two 4×4 reps of part *a*); $M^2 \not\equiv 0$ for the 6×6 rep of part *a*, but is undiagonalizable ([11]).

One might change his model of « particle » by choosing a different complete set from the generator algebra in an attempt to get a mass spectrum. One can include the four space-time momenta (here M_{5m}) as is done ([12]) in extracting the ∞-dimensional unitary reps of L_{inh}. Reps are then obtained which are the direct generalization of the Bargmann-Wigner reps of the L_{inh} generator algebra. For example, for spin 0 we get ([13])

$$
(5)\qquad \left\{\begin{aligned}
&M_{5m} \to \quad p_m \qquad\qquad (p_m \text{ numbers}, -\infty \leqslant p_m \leqslant +\infty),\\
&M_{mn} \to \quad \frac{1}{i}\left(p_m \frac{\partial}{\partial p^n} - p_n \frac{\partial}{\partial p^m}\right),\\
&M_{0m} \to -\frac{1}{2}\, p_m \Box_p^2 - i\Lambda \frac{\partial}{\partial p^m},\\
&M_{50} \equiv i\Lambda \to i\left(p_m \frac{\partial}{\partial p_m} + 2\right).
\end{aligned}\right.
$$

Here $p^m = g^{mn} p_n$ g_{mn} is the Lorentz metric of signature $(+++-)$, and $\Box_p^2$ is the d'Alembertian with respect to the p_m. It can be easily verified that (5) satisfies the commutator algebra of the 6-dimensional rotation group G. The first two lines define a Bargmann-Wigner spin 0 rep of L_{inh} if $m^2 \equiv (p_4)^2 - \boldsymbol{p}^2$ is some constant $\geqslant 0$. One gets a different rep of L_{inh} for each such m. Each of these—defined by (5) and a given m—*is* a rep of G also. In fact, all these reps are equivalent, they define one and the same rep of G. This is proved by substituting (5) into the expressions Q, R, W of the center and noting that one gets ([13]) 0, 0, 4 respectively, independent of the value of m. The same thing proves that any of these reps is equivalent to the rep defined by (5) alone with m^2 having any spectrum, discrete or continuous, desired. Hence it is impossible to get a significant bare particle mass spectrum in this way either.

This is perhaps not surprising in a theory in which length, and hence mass, is not invariant under the whole symmetry group. Doubtless the most radical

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([11]) This is related to the fact that these reps are unitary relative to the *indefinite* metrics of the representation spaces. E.g., the 6×6 rep is unitary relative to $G_{\mu\nu}$ of signature $(+++-+-)$.

([12]) V. Bargmann and E. Wigner: *Proc. Nat. Acad. Sci.* (*USA*), **34**, 211 (1948).

([13]) This is due essentially to Y. Murai, given in *Prog. Theor. Phys.*, **11**, 441 (1954).

innovation which would be introduced into physics by the use of this symmetry group would be in our concept of elementary particle mass.

d) Already mentioned above is the fact that the reps of G yield, not single conventional particles, but finite families of these. The reps of small dimension yield families of small membership. For example it is easily seen from the remarks in part *a* that the rep of the α-sequence defined by p has

$$n(p) \equiv \sum_{a=\mathrm{Min}(j_1,j_2)} (2a+1) = \begin{cases} (p+1)^2, & p = \text{integer} \\ (p+\frac{1}{2})(p+\frac{3}{2}), & p = \text{half odd integer} \end{cases}$$

member particles, the sum being over all j_1, j_2 such that $j_1+j_2=p$. Thus $n(\frac{1}{2})=2$, $n(1)=4$, $n(\frac{3}{2})=6$, etc. The 6-dimensional rep mentioned in part *a* has four particles, and so on. These particles would be expected to be grouped in nature by some overall similarity or similiarities. This will be discussed further in the last section.

3. – Some concrete examples.

It may be useful in clarifying this unfamiliar mathematics to show explicitly the derivation of these results on several of the simpler reps. We choose for this purpose the pair of reps of least dimension, the two 4-dimensional *spinor* reps, and the 6-dimensional *vector* rep.

1) *The spinor reps.* – These can be built from the six 4×4 spinor matrices Π_μ ($\mu=0,1,\ldots,5$) satisfying the « anticommutation » relation

$$\bar{\Pi}_\mu \Pi_\nu + \bar{\Pi}_\nu \Pi_\mu = 2G_{\mu\nu}\mathbf{1}\,, \tag{6}$$

where $\mathbf{1}$ means the 4×4 unit matrix and bar means complex conjugation. This Π-algebra is a higher-dimensional analogue of the spinor algebra satisfied by the four 2×2 matrices σ_1, σ_2, σ_3 and $\pm i\mathbf{1}$ with the Lorentz metric on the right. It is *not* the type of spinor algebra satisfied by the four 4×4 Dirac γ's (note the complex conjugation in (6)!) There are two inequivalent reps of the Π-algebra, both 4×4; a convenient realization for our purposes is

$$\begin{cases} \Pi_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \Pi_2 = i\mathbf{1}\,, \quad \Pi_3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \\ \Pi_4 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \Pi_0 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \Pi_5 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix} \end{cases} \tag{7}$$

and another rep in which the sign of Π_2 in (7) is changed. Remember that we are using the non-principal axes form (1). The generators of the spinor reps of G are then defined by $M_{\mu\nu} = (1/2i)\bar{\Pi}_{[\mu}\Pi_{\nu]}$. We get for the spin and Δ ope-

rators

$$s_x = \frac{1}{2}\begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad s_y = \frac{1}{2}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad s_z = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$

$$\boldsymbol{s}^2 = \frac{3}{4}\mathbf{1}, \quad \Delta = \frac{1}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

where the **1**'s in Δ are 2×2 unit matrices, and another array in which the signs of s_y and s_z are changed. Hence as stated above, each of these reps describes a pair of spin $\frac{1}{2}$ particles differentiated by $\delta = \pm\frac{1}{2}$. The operator R (formula (4)) evaluated with the matrices (7) is

$$R = \frac{8}{(2i)^3}\overline{\Pi}_1\Pi_2\overline{\Pi}_3\Pi_4\overline{\Pi}_5\Pi_0 = 4s_z\overline{\Pi}_3\Pi_4\Delta = +\mathbf{1},$$

while the other spinor rep has $R = -\mathbf{1}$. Finally, the mass squared operator $M^2 \equiv 0$ here, since it involves $\overline{\Pi}_5\Pi_5 \equiv 0$.

The single 8×8 spinor rep of the entire orthogonal group G (that is, including the improper transformations) just comprises these two 4×4 reps of the *proper* subgroup of G. The particle pairs belonging to these two reps thus stand in the same relation as, say, the « large » and « small » components of a Dirac 4-spinor, which transform under the two inequivalent 2×2 reps of the *proper* Lorentz group.

2) *The vector rep.* – The formula for the matrix elements of the generators of this rep can be derived directly from the form $\mathbf{1}+\boldsymbol{\epsilon}$ (an infinitesimal 6×6 rotation matrix) in the same way that this is done for the 3×3 rep of the ordinary rotation group. One gets

$$(M_{\mu\nu})^a{}_b = \frac{1}{i}\left(\delta^a_\mu G_{\nu b} - \delta^a_\nu G_{\mu b}\right) \qquad (a, b = 0, 1, \ldots, 5).$$

This gives

$$(M^2_{ij})^a{}_b = \delta^a_i\delta^i_b + \delta^a_j\delta^j_b \qquad (i, j = 1, 2, 3 \text{ unsummed})$$

and hence $\boldsymbol{s}^2$ and Δ are the matrices

$$(\boldsymbol{s}^2)^a{}_b = 2(\delta^a_1\delta^1_b + \delta^a_2\delta^2_b + \delta^a_3\delta^3_b),$$

$$\boldsymbol{s}^2 = 2\left(\begin{array}{c|c|c} 0 & & \\ \hline & 1 & \\ & 1 & \\ & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right)$$

and

$$\Delta^{n}{}_{b} = \delta^{n}_{0} G_{5b} - \delta^{n}_{5} G_{0b} ,$$

$$\Delta = \begin{pmatrix} -1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}$$

All off diagonal places in these matrices are understood to be occupied by zeroes. Therefore we see, as maintained above, that this rep contains one spin 1 particle corresponding to $\delta = 0$ and three spin 0 particles differentiated by $\delta = +1,\ 0,\ -1$. The quantum number r for this rep vanishes:

$$R^{a}_{d} = -\frac{8}{6!}\left(\frac{2}{i}\right)^{3} \varepsilon^{\mu\nu\lambda\xi\varrho\tau}\, \delta^{a}_{\mu} G_{\nu b}\, \delta^{b}_{\lambda} G_{\xi c}\, \delta^{c}_{\varrho} G_{\tau d} ,$$

$$= -\frac{8}{6!}\left(\frac{2}{i}\right)^{3} \varepsilon^{\mu\nu\lambda\xi\varrho\tau}\, \delta^{a}_{\mu} G_{\nu\lambda} G_{\xi\varrho} G_{\tau d} = 0 ,$$

by the symmetry of $G_{\nu\lambda}$. Hence it is a singlet rep.

The mass squared operator comes out to be the undiagonalizable matrix

$$M^{2} = 4 \begin{pmatrix} & & \\ & & \\ & & \\ & & \\ 1 & & \end{pmatrix}$$

where again all empty places are understood to be occupied by zeroes.

4. – Some physical remarks.

It is seen that the symmetry group G produces another angular momentum (say $\boldsymbol{I}$) besides the particle's spin; with our choice of complete set of commuting observable Δ is its z-component. This angular momentum operator is « ultra-Lorentz » in having nothing to do with the generation of Lorentz transformations; it is thus in this respect like the isotopic spin $\boldsymbol{T}$. The theory

presents a definite scheme for assigning values of this ultra-Lorentz spin to the particles contained in a rep, for example, for the α-sequence it is simply (3). The spin Δ even has the right sort of eigenvalues for the simple reps discussed here: for the two spinor reps $I = \frac{1}{2}$ and $\delta = I_z' = \pm\frac{1}{2}$, which suggests the nucleon and antinucleon; for the vector rep $I = 1$, $\delta = I_z' = 1, 0, -1$ for the three spin 0 particles, which makes one think of the pions. (Of course there is also the spin 1 particle with $I = \delta = 0$.)

The rep structure of G, consisting either of singlets or pairs of reps in doublets, suggests the division of known particles into those particle families whose antiparticles form the same family (photon, π-meson, ...) and those whose antiparticles form different families (nucleons, K-mesons, ...). Since the member reps of a doublet are just interchanged by an improper transformation of G, this suggests that particle conjugation might be assimilated to the group of co-ordinate transformations in this way. One knows in fact from the $C\mathcal{P}\mathcal{T}$ theorem that it has a connection with improper transformations of the co-ordinates. We should remark in this place also that G is, properly speaking, a group of projective transformations, so that phase and gauge transformations, which serve to define the charge operator, might conceivably be found already in it.

The particle families defined by the reps of G could possibly account for the empirical grouping of the known particles into families which we acknowledge by giving the member particles a common generic name: the nucleon (n, p), the K-meson (K^+, K^0, etc.), the Σ hyperon (Σ^+, Σ^0, Σ^-). A necessary condition for a group of particles to form one of these families is that they have the same or closely grouped masses. Certainly the member particles of a rep of G would be expected to fulfill this requirement, any spread in real masses coming from normally small virtual cloud effects.

The real masses of these particles, as typical *dynamical* quantum numbers, would have to be calculated from a correct quantized field theory of these various fields in interaction, in a way unknown to us at present. The discussion of Sect. 2, part *c*, makes it seem very probable that the use of G as symmetry group would force a revision in the traditional way of handling mass theoretically.

One can still make good headway in investigating the physical validity of a group such as G even if one bypasses the difficult questions like that of mass which can only be resolved if one has a correct dynamical theory of interacting fields. One can concentrate on the properties which are inherent in the bare-particle theories or in the *c*-number theories of the interacting fields. Such properties are, we can be thankful, definitely not scarce or of minor interest. An example of the first is the set of « true » quantum numbers, which we have been investigating in this paper. As an example of the second, consider one of the 4×4 spinor reps given explicitly in Sect. 3. If one hypothesized

that this was the nucleon, one could try to write down a correct c-number wave equation for this field in interaction with the electromagnetic field. Then one could immediately see whether the magnetic moments of proton and neutron differed in magnitude and whether, in fact, they were roughly the experimental values. Indeed, the theoretical prediction of the electron magnetic moment was one of the main props of the c-number Dirac theory before its triumphs as a quantized field.

RIASSUNTO (*)

Si esaminano i tipi dei numeri quantici delle particelle elementari quali risultano dall'estensione di L_{inh} (gruppo di Lorentz inomogeneo) a un più vasto gruppo di simmetria che non è semplicemente $L_{\text{inh}} \times$ rotazioni isotopiche. I risultati principali sono: *a*) esiste un altro spin, non avente nulla in comune con le trasformazioni di Lorentz, del tipo dello spin isotopico; *b*) ogni rappresentazione irriducibile del gruppo comprende una famiglia finita di particelle convenzionali con un certo spettro di spin; lo spettro della componente z di questo spin « ultra-lorentziano » è connesso in modo definito allo spettro dello spin; *c*) si trova una struttura simile alla divisione empirica di tutte le famiglie di particelle nella classe di quelle che coincidono con le famiglie delle loro antiparticelle e nella classe di quelle le cui antiparticelle formano famiglie differenti; *d*) è implicita la revisione del concetto di massa della particella nuda.

(*) *Traduzione a cura della Redazione.*

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On a Possibility for the Elimination of the Non-Physical Consequences of the Indefinite Metric.

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Summary. — Bogoljubov's method for the elimination of the non-physical consequences of the indefinite metric has been analised. We have proved that this method is equivalent to a very complicated non-local theory in which, however, there is no need to use the indefinite metric and where the usual concepts of a quantum theory can be applied. Besides, we have outlined the way of constructing a non-local theory having usual commutation relations.

1. – Introduction.

According to the recent development of the quantum field theory it is very likely that the employment of the indefinite metric is unavoidable also from several different points of view. In this case the whole Hilbert space H of the state vectors consists of two (orthogonal) subspaces H_p and H_n. H_p contains the physical states and H_n contains the non-physical ones. The state vectors of H_p have positive norm, while those of H_n may admit both negative and positive norm. Thus every state vector has the form

$$(1) \quad \Psi = \Psi_p + \Psi_n = P_p\Psi + P_n\Psi\,, \qquad P_p + P_n = 1, \qquad P_p P_n = P_n P_p = 0\,.$$

where the P-s are projection operators. They project the state vectors into the subspaces H_p and H_n. Ψ_n however may lead to non-physical results, *e.g.* negative probabilities may occur.

In quantum electrodynamics in the method of Gupta and Bleuler [1] dif-

[1] S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

ferent Ψ having common Ψ_p describe the same physical state. The Lorentz condition assures that the non-physical part Ψ_n does not contribute to the mean value of the physical observables and has the norm zero.

Heisenberg's theory [2] is stated to be constructed so that if $\Psi(-\infty) = \Psi_p(-\infty)$ then $\Psi(+\infty)$ belongs also to H_p. This happens indeed in the case of the dipole ghost state of the Lee model as discussed by HEISENBERG [3].

$\Psi(+\infty)$ does not belong, however, to H_p in the case of the discrete ghost state analised by KÄLLÉN and PAULI [4], therefore this can not be regarded as a physically acceptable theory. ASCOLI and MINARDI [5] have shown in general that a theory in which the total Hamiltonian has eigenvectors with negative norm can not be used for the description of real phenomena without further modification.

If we demand the usual probabilistic interpretation of the state vectors only at infinity ($t = \pm\infty$), physically acceptable results can be obtained by means of appropriate modifications, as it has been pointed out by BOGOLJUBOV, MEDVEDEV and POLIVANOV [6]. They suppose that the non-physical part Ψ_n of the state vector is uniquely determined by the physical part Ψ_p.

In the following we give the « axiomatics » of this method. On a special model we show that this method is equivalent to a very complicated non-local theory, in which, however, we can work with the usual concepts of a quantum theory. Thus the rather strange modifications in the basic concepts seem to be only mathematical tools for handling a complicated but otherwise ordinary quantum theoretical problem. Furthermore the construction of non-local theories on similar lines will be mentioned.

2. – The physical $\tilde{S}$-matrix.

We suppose that the following assumptions are valid:

1) *Only $P_p \alpha P_p = \alpha$-type operators can be measured long before and after the scattering (at $t = \pm\infty$).*

Here it is naturally supposed that the interaction is adiabatically switched on and off at infinity.

Denote by α a complete system of observables in H_p; $\alpha \Psi_{pi} = a_i \Psi_{pi}$. Then the Ψ_{pi} form a complete (orthogonal) system of state vectors in H_p. They can be normalized to $+1$.

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(2) W. HEISENBERG: *Rev. Mod. Phys.*, **29**, 269 (1957).
(3) W. HEISENBERG: *Nuc. Phys.*, **4**, 532 (1957).
(4) G. KÄLLÉN and W. PAULI: *Mat. Fys. Medd. Dan. Vid. Selsk.* **30**, n. 7 (1955).
(5) R. ASCOLI and E. MINARDI: *Nuovo Cimento*, **8**, 951 (1958).
(6) N. N. BOGOLJUBOV, B. V. MEDVEDEV and M. K. POLIVANOV: preprint of *J.I.N.R.* (1958); *CERN Conference Reports*, **129**, (1958).

2) *The possible results of the measurements of α are its eigenvalues. If measuring α we obtain the value a_i, the state of the physical system is described by*

$$\Psi_i = (P_p + N)\Psi_{pi}\,; \qquad P_n N P_n = N\,. \tag{2}$$

N is the operator which determines the Ψ_n belonging to the physical part Ψ_p.

The norm of the Ψ_i-s may be positive, negative or zero. They are normalized expediently so that $\|\Psi_{pi}\| = +1$. In this case the mean value of α, $\overline{\alpha} = (\Psi_i, \eta\alpha\Psi_i)$ equals just a_i. Here η is the metric operator $\eta^2 = 1$, $\eta = \eta^*$.

According to (2) the initial state of the system is

$$\Psi(-\infty) = (P_p + N)\Psi_p(-\infty)\,, \tag{3}$$

from which

$$\Psi(+\infty) = S\,\Psi(-\infty) = S(P_p + N)\Psi_p(-\infty)\,, \tag{4}$$

$$\Psi_p(+\infty) = P_p\Psi(+\infty) = P_p S(P_p + N)\Psi_p(-\infty)\,. \tag{5}$$

Here S is the S-matrix naturally.

The matrix

$$\tilde{S} = P_p S(P_p + N) \tag{6}$$

connecting the physical parts is called the physical $\tilde{S}$ matrix.

3) *The probability of obtaining the value a_j when we measure α in the state Ψ* is $c_j^* c_j$, where $c_j = (\Psi_{pj}, \eta\Psi) = (\Psi_{pj}, \Psi)$.

Thus the transition probability amplitude $i \to j$ is

$$c_{ij} = (\Psi_{pj}, \Psi(+\infty)) = (\Psi_{pj}, \Psi_p(+\infty)) = (\Psi_{pj}, \tilde{S}\Psi_{pi})\,. \tag{7}$$

This is consistent only if

$$\sum c_j^* c_j = \|\Psi_p(+\infty)\| = 1 = \|\Psi_p(-\infty)\|\,,$$

i.e.

$$\tilde{S}^+\tilde{S} = \tilde{S}\tilde{S}^+ = P_p\,. \tag{8}$$

N has to be determined just from this requirement.

There is a further requirement on N. The conservation theorems must be fulfilled; *i.e.* if $A_p = P_p A P_p$ is a physical observable which conserves it is necessary that

$$[\tilde{S}, A_p] = 0\,. \tag{9}$$

It is easily seen, that requirements (8) and (9) do not determine N uniquely. Indeed to every $\widetilde{S}$ satisfying (8) and (9) corresponds from (6) one N. Here we might use the formulae

$$(10)\qquad \begin{cases} P_p S^+ P_p = (P_p S P_p)^{-1}(P_p - P_p S P_n (P_n S P_n)^{-1} P_n S P_p (P_p S P_p)^{-1})^{-1}\,, \\ P_p S^+ P_n = -\,P_p S^+ P_p P_p S P_n (P_n S P_n)^{-1}\,, \end{cases}$$

and similar expressions changing $p \to n$ (this follows from $S^+ S = 1$), and

$$(11)\qquad [P_p S P_p, A_p] = 0\,, \qquad P_n S P_p A_p = A_n P_n S P_p\,, \qquad P_p S P_n A_n = A_p P_p S P_n\,.$$

Here $A_p = P_p A P_p$, $A_n = P_n A P_n$, $[S, A] = 0$ and in consequence of the adiabatic switching off $P_p A P_n = P_n A P_p = 0$.

A trivial possibility is $\widetilde{S} = P_p$. Then $N = (P_p S P_n)^{-1}(P_p - P_p S P_p)$.

Bogoljubov proposes something similar in the H_n space. Defining $\Psi_n(+\infty) = \widetilde{S}_n \Psi_n(-\infty)$ from (3) and (4) we obtain

$$\widetilde{S}_n = P_n S (P_n + N^{-1})\,.$$

Supposing $\widetilde{S}_n = -P_n$ ($\widetilde{S}_n^+ \widetilde{S}_n = P_n$; choosing the negative sign we obtain that the physical reactance matrix is $\widetilde{K} = P_p K P_p$) this gives

$$(12)\qquad N = -\,(P_n + P_n S P_n)^{-1} P_n S P_p\,,$$

and from (6)

$$(13)\qquad \widetilde{S} = P_p S (P_p - (P_n + P_n S P_n)^{-1} P_n S P_p)\,.$$

(8) and (9) are naturally satisfied.

We may require that also $\Psi(+\infty)$ should satisfy (3), similarly to the case of $\Psi(-\infty)$:

$$(14)\qquad \Psi(+\infty) = (P_p + N)\Psi_p(+\infty)\,,$$

from which

$$(15)\qquad N\widetilde{S} = P_n S (P_p + N)\,.$$

It can be seen however on examples that $\widetilde{S}$ obtained from (6) and (15) does not satisfy in general (8).

The simplest non-trivial (however non-linear) $\widetilde{S}$ results from

$$(16)\qquad N\Psi_p(-\infty) = (P_p S P_n)^{-1}(a - 1) P_p S P_p \Psi_p(-\infty)\,,$$

$$a^{-2} = \| P_p S P_p \Psi_p(-\infty) \|\,,$$

whence

$$\bar{S}\Psi_p(-\infty) = aP_pSP_p\Psi_p(-\infty)\,. \tag{17}$$

The above considerations can always be applied if there is no other possibility for the elimination of the non-physical consequences. Thus it can be used also in the Lee model. Regarding those states in which one or more bare V particles are present as non-physical ones, we may obtain for the N-θ interaction a physically interpretable result.

This method proved to be most powerful in case of theories which offer a Pauli-Villars (7) type of regularization. In the following such a model will be outlined.

3. – Application to a model.

For the sake of simplicity let us treat the situation found in the case of interacting fermion and neutral scalar meson fields (8) with the interaction Hamiltonian

$$H(x) = g\,\overline{\psi}(x)\psi(x)\varphi(x)\,. \tag{18}$$

The fermion field is a superposition of the true (observable) fermion field ψ_0 and several (already two are sufficient) fictious fermion fields:

$$\psi(x) = \psi_0(x) + \sum c_n\psi_n(x)\,, \tag{19}$$

having the commutation relations:

$$\{\psi_0(x), \overline{\psi}_0(x')\} = -\,iS(x-x')\,, \qquad \{\psi_n(x), \overline{\psi}_n(x')\} = -\,i\varepsilon_n S_n(x-x')\,. \tag{20}$$

Similarly the boson field has the form (one fictitious field is sufficient)

$$\varphi(x) = \varphi_0(x) + \sum b_n\varphi_n(x)\,, \tag{21}$$

with the commutations relations

$$[\varphi_0(x), \varphi_0(x')] = i\hbar c\Delta(x-x')\,, \qquad [\varphi_n(x), \varphi_n(x')] = i\hbar c\varrho_n\Delta_n(x-x')\,. \tag{22}$$

and ε_n, ϱ_n have the value ± 1. The coefficients satisfy the usual Pauli-

(7) W. Pauli and F. Villars: *Rev. Mod. Phys.*, **21**, 434 (1949).
(8) B. V. Medvedev and M. K. Polivanov: preprint of *J.I.N.R.* (1958).

Villars conditions

$$(23)\qquad \begin{cases} 1+\sum \varepsilon_n c_n^* c_n = 0\,, \quad \dots\,, & (m^2)^s+\sum (m_n^2)^s \varepsilon_n c_n^* c_n = 0\,, \quad \dots\,, \\ 1+\sum \varrho_n b_n^2 = 0\,, \quad \dots\,, & (\mu^2)^s+\sum (\mu_n^2)^s \varrho_n b_n^2 = 0\,, \quad \dots\,. \end{cases}$$

From such a model using $\widetilde{S}$ as defined above, we may get a regular theory, where keeping m and μ finite the self energy, etc., will be also finite. MEDVEDEV and POLIVANOV [8] have calculated the self energy of a fermion in the above model using $\widetilde{S}$ as defined by (13). Calculations will be most simple using (17) as $\widetilde{S}$. In this case we obtain (apart from a) the usual S-matrix elements using everywhere the regularized causal functions in them.

Now we show, how the above theory is equivalent to a non-local, but otherwise ordinary theory.

Only the quantities of the type $P_p \alpha P_p = \alpha$ are observables. At the time t their mean values are

$$(24)\qquad (\Psi(t), \eta\alpha(t)\Psi(t)) = (\Psi_p(t), \alpha(t)\Psi_p(t)) = (\Psi_p(-\infty), \widetilde{U}^+(t)\alpha(t)\widetilde{U}(t)\Psi_p(-\infty))\,,$$

where (cf. (3))

$$(25)\qquad \widetilde{U}(t) = P_p U(t)(P_p + N)\,.$$

Naturally $\widetilde{U}^+\widetilde{U} = P_p$ is valid only at $t = \pm\infty$. The « physical » Heisenberg representation operators

$$(26)\qquad \hat{\alpha}(t) = \widetilde{U}^+(t)\alpha(t)\widetilde{U}(t)\,,$$

are operating now in H_p. H_n and the indefinite metric etc., are not necessary, the state is uniquely determined by $\Psi_p(-\infty)$.

Choosing $\alpha = \psi_0(x)$, we obtain

$$(\gamma_\mu \partial_\mu + \varkappa)\hat{\psi}_0(x) = \gamma_4 \widetilde{U}^+(t)\left\{\widetilde{U}^{+-1}(t)\,\partial_4 \widetilde{U}^+(t)\,\psi_0(x) + \psi_0(x)\,\partial_4 \widetilde{U}(t)\,\widetilde{U}^{-1}(t)\right\}\widetilde{U}(t)\,,$$

the right-hand side shows the non-local character of the theory. In the case of the above model when $c_n = 0$ and all Bose-fields are regarded as fictive ones, and using N from (12) we obtain

$$(27)\qquad (\gamma_\mu \partial_\mu + \varkappa)\hat{\psi}_0(x) = -\frac{g^2}{\hbar c}\int_{-\infty}^{+\infty} \hat{\psi}_0(x)\varepsilon(x-x')K(x-x')\,\hat{\bar{\psi}}_0(x')\,\hat{\psi}_0(x')\,\mathrm{d}x' -$$

$$-\frac{g^2}{\hbar c}\int_{-\infty}^{+\infty}\int \hat{\bar{\psi}}_0(x'')\,\hat{\psi}_0(x'')\varepsilon(x-x')K(x''-x')\,i\gamma_4 S(x-x')\,\hat{\psi}_0(x')\,\mathrm{d}x'\,\mathrm{d}^3x'' +$$

$$+ \text{higher order terms in } g\,.$$

Here

$$K(x) = \tfrac{1}{2} \sum b_n^2 \varrho_n \Delta_n^{(+)}(x) ,$$

$$x_0'' = x_0 .$$

The commutation relations of $\hat{\psi}$ have the simple form (20) naturally only at $x_0 = x_0' = \pm \infty$.

At this point we may forget about Bogoljubov's method, and by means of similar model theories using a $\tilde{U}(t)$ which fulfills the requirement $\tilde{U}^+(t)\tilde{U}(t) = P_p$, we can construct a non-local theory. Such a $\tilde{U}(t)$ on the analogy of (13) is *e.g.*

$$\tilde{U}(t) = P_p U(t)(P_p - (P_n + P_n U(t) P_n)^{-1} P_n U(t) P_p) .$$

Defining similarly $\hat{\alpha}$ as

$$\hat{\alpha} = \tilde{U}^+(t)\alpha(t)\tilde{U}(t)$$

for $\hat{\psi}$ we obtain

$$\text{(28)} \qquad \begin{cases} (\gamma_\mu \partial_\mu + \varkappa)\hat{\psi}(x) = \gamma_4 \tilde{U}^+(t) \left[\tilde{U}(t)\, \partial_4 \tilde{U}^+(t),\, \psi(x) \right] \tilde{U}(t) , \\ \left\{ \hat{\psi}(x),\, \hat{\bar{\psi}}(x') \right\} = -\, i S(x - x') \qquad \text{if } (x_\mu - x'_\mu)^2 \geqslant 0. \end{cases}$$

From (28) we obtain an equation in the case of the above model similar to (27) (naturally with an other kernel) but with two extra terms originating from $\partial_4 N(t)$.

RIASSUNTO (*)

Si è analizzato il metodo di Bogoljubov per l'eliminazione delle conseguenze non fisiche della metrica indefinita. Abbiamo dimostrato che questo metodo è equivalente a una teoria non locale molto complicata, in cui, tuttavia, non occorre far uso della metrica indefinita e sono applicabili i concetti usuali di una teoria quantistica. Abbiamo inoltre delineato il modo di costruire una teoria non locale dotata di relazioni di commutazioni ordinarie.

(*) *Traduzione a cura della Redazione.*

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The Born Expansion in Non-Relativistic Quantum Theory.

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Summary. — Potential scattering of a non-relativistic particle is studied by means of the Born series for the Green's function, wave function, and scattering amplitude. For a wide class of potentials, these series are proved to converge at sufficiently high incident energies. The validity of the Born approximation in the high energy limit is also shown.

1. – Introduction.

The application of the Born approximation and of the iterated Born series to scattering problems in quantum mechanics [1] is widely known. Although the method's accuracy is recognized to improve with increasing energy, many questions relating to the convergence of the expansion remain unclarified. Progress has been made by Jost and Pais [2] who established convergence for sufficiently weak spherically symmetric potentials and showed, by a specific example, that the Born series for the scattering amplitude may indeed diverge under conditions of practical importance.

The attention of the present authors was drawn to this problem by their study of dispersion relations for potential scattering. A knowledge of the scattering amplitude in the high energy limit is required and the investigation is simplified considerably if use of the Born series is permissible. One may

[1] M. Born: *Zeits. f. Phys.*, **38**, 803 (1926).

[2] R. Jost and A. Pais: *Phys. Rev.*, **82**, 840 (1951). See, also, W. Kohn: *Rev. Mod. Phys.* **26**, 292 (1954).

also hope that understanding of the Born series in a comparatively simple context will supply a modest signpost in the wilderness of the field theoretic problem.

In this note, we study a non-relativistic particle scattered by a potential. We show that, for a general class of potentials, the Born series for the Green's function always converges for sufficiently high energies, and in the high energy limit, the series is approximated by its leading term. The treatment is then extended to wave functions and scattering amplitudes.

2. – Preliminary remarks.

In the integral formulation of the scattering problem, a wave function $\psi(\boldsymbol{r})$ is sought which satisfies [3]

$$\psi = \psi_0 + G_0 V \psi \,, \tag{2.1}$$

$$= \psi_0 + G V \psi_0 \,, \tag{2.2}$$

where the (outgoing wave) Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ obeys

$$G = G_0 + G_0 V G \,. \tag{2.3}$$

The free particle functions are given by

$$\psi_0(\boldsymbol{r}) = \exp[i\boldsymbol{k}\cdot\boldsymbol{r}] \,, \tag{2.4}$$

$$G_0(\boldsymbol{r}, \boldsymbol{r}') = -(4\pi)^{-1} |\boldsymbol{r} - \boldsymbol{r}'|^{-1} \exp[ik|\boldsymbol{r} - \boldsymbol{r}'|] \,, \tag{2.5}$$

where the wave number k is related to the incident energy by $E = \hbar^2 k^2/2m$. Let units be chosen so that $\hbar = 1$; then k also represents the particle momentum.

The Born series are defined by iteration of (2.1) and (2.3):

$$\psi = \sum_{n=0}^{\infty} \psi_n \,, \tag{2.6}$$

$$G = \sum_{n=0}^{\infty} G_n \,, \tag{2.7}$$

where

$$\psi_{n+1}(\boldsymbol{r}) = \int G_0(\boldsymbol{r}, \boldsymbol{s}) V(\boldsymbol{s}) \psi_n(\boldsymbol{s}) \, d\boldsymbol{s} \tag{2.8}$$

---

[3] The potential for these equations is $(\hbar^2/2m)V(\boldsymbol{r})$, but for brevity, we shall call $V(r)$ itself the potential.

and

$$G_{n+1}(\boldsymbol{r}, \boldsymbol{r}') = \int G_0(\boldsymbol{r}, \boldsymbol{s}) V(\boldsymbol{s}) G_n(\boldsymbol{s}, \boldsymbol{r}') \, \mathrm{d}\boldsymbol{s} \, .$$

From the last equation, the more general relation

$$G_{n+m+1}(\boldsymbol{r}, \boldsymbol{r}') = \int G_n(\boldsymbol{r}, \boldsymbol{s}) V(\boldsymbol{s}) G_m(\boldsymbol{s}, \boldsymbol{r}') \, \mathrm{d}\boldsymbol{s} \tag{2.9}$$

follows directly. It is useful to introduce an associated set of functions:

$$\begin{cases} g_0(\boldsymbol{r}, \boldsymbol{r}') = 1 \, , \\ g_n(\boldsymbol{r}, \boldsymbol{r}') = G_n(\boldsymbol{r}, \boldsymbol{r}') / G_0(\boldsymbol{r}, \boldsymbol{r}') \, , \\ g\ (\boldsymbol{r}, \boldsymbol{r}') = G\ (\boldsymbol{r}, \boldsymbol{r}') / G_0(\boldsymbol{r}, \boldsymbol{r}') \, , \end{cases} \tag{2.10}$$

and to consider, in place of (2.7), the series

$$g(r, r') = \sum_{n=0}^{\infty} g_n(r, r') \, . \tag{2.11}$$

With the notation

$$D_{r,r'}(s) = \frac{1}{4\pi} \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{|\boldsymbol{r} - \boldsymbol{s}| \, |\boldsymbol{r}' - \boldsymbol{s}|} \exp\left[ik(|\boldsymbol{r} - \boldsymbol{s}| + |\boldsymbol{r}' - \boldsymbol{s}| - |\boldsymbol{r} - \boldsymbol{r}'|)\right] , \tag{2.12}$$

Eq. (2.9) assumes the form

$$g_{n+m+1}(\boldsymbol{r}, \boldsymbol{r}') = \int g_n(\boldsymbol{r}, \boldsymbol{s}) D_{r,r'}(\boldsymbol{s}) V(\boldsymbol{s}) g_m(\boldsymbol{s}, \boldsymbol{r}') \, \mathrm{d}\boldsymbol{s} \, . \tag{2.13}$$

A « norm » may be assigned to the g functions as follows:

$$\|g_i\| = \operatorname*{Max}_{r,r'} |g_i(\boldsymbol{r}, \boldsymbol{r}')| \tag{2.14}$$

i.e., the norm $\|g_i\|$ is the maximum numerical value attained by $g_i(\boldsymbol{r}, \boldsymbol{r}')$ as $\boldsymbol{r}$ and $\boldsymbol{r}'$ vary. The norm is well defined for any function of two variables and is either a non-negative real number or infinity.

A corresponding numerical measure of the potential may be expressed in terms of a dimensionless ([3]) quantity $I_V(r)$:

$$I_V(\boldsymbol{r}) = \int |\boldsymbol{r} - \boldsymbol{s}|^{-1} |V(\boldsymbol{s})| \, \mathrm{d}\boldsymbol{s} . \tag{2.15}$$

The « strength » $\|V\|$ of the potential V is then defined by

$$\|V\| = (2\pi)^{-1} \operatorname*{Max}_{r} I_r(\boldsymbol{r}) . \tag{2.16}$$

3. – The Born series for simple dotentials.

A potential $V(\boldsymbol{s})$ will be termed « simple » if both V and its gradient exist for all $\boldsymbol{s}$, are bounded in magnitude, and vanish when $|\boldsymbol{s}|$ exceeds some finite magnitude S. The present section is concerned only with simple potentials; the extension of results to a more general class of potentials is considered later.

The strength $\|V\|$ of a simple potential is certainly finite, and one may distinguish between weak and strong potentials according as to whether $\|V\| < 1$ or $\|V\| \geqslant 1$.

We begin the mathematical argument with the observation that

$$\operatorname*{Lim}_{k\to\infty} \|g_1\| = 0 , \tag{3.1}$$

Equation (3.1) is stronger than the statement

$$\operatorname*{Lim}_{k\to\infty} g_1(\boldsymbol{r}, \boldsymbol{r}') = 0 \qquad \text{for each } \boldsymbol{r}, \boldsymbol{r}'$$

which is merely a three dimensional analogue of the Rieman-Lebesgue lemma. Its proof is given in the Appendix.

The triangle inequality

$$|\boldsymbol{r} - \boldsymbol{r}'| \leqslant |\boldsymbol{r} - \boldsymbol{s}| + |\boldsymbol{s} - \boldsymbol{r}'|$$

implies an inequality for $D_{r,r'}(\boldsymbol{s})$:

$$|D_{r,r'}(\boldsymbol{s})| \leqslant \frac{1}{4\pi} |\boldsymbol{r} - \boldsymbol{s}|^{-1} + \frac{1}{4\pi} |\boldsymbol{r}' - \boldsymbol{s}|^{-1} . \tag{3.2}$$

Hence, by (2.13) and (2.14),

$$\|g_{n+m+1}\| \leqslant \|g_n\| \, \|g_m\| \operatorname*{Max}_{r,r'} \frac{1}{4\pi} \int (|\boldsymbol{r} - \boldsymbol{s}|^{-1} + |\boldsymbol{r}' - \boldsymbol{s}|^{-1}) \, |V(\boldsymbol{s})| \, \mathrm{d}\boldsymbol{s} \leqslant \|g_n\| \, \|g_m\| \, \|V\| . \tag{3.3}$$

Eq. (2.11) may now be treated by comparison with a geometric series. For example, when $\|V\| < 1$, (3.3) tells us that

$$\|g_n\| \leqslant \|g_0\| \, \|V\|^n = \|V\|^n . \tag{3.4}$$

It follows that for such potentials, (2.11) converges absolutely and uniformly in r, r', and that

$$\|g\| \leqslant \sum_n \|V\|^n = (1 - \|V\|)^{-1}. \tag{3.5}$$

The absolute and uniform convergence of (2.7) is also implied, except in a spatial neighborhood of the singular points $r = r'$.

Now even if V is not weak, Eq. (3.1) shows that for all sufficiently large k, we have $\|g_1\| \, \|V\| < 1$.

Since, by (3.3)

$$\|g_{2n}\| \leqslant (\|g_1\| \, \|V\|)^n \tag{3.6}$$

and

$$\|g_{2n+1}\| \leqslant (\|g_1\| \, \|V\|)^n \|g_1\|, \tag{3.7}$$

we conclude that, for all sufficiently large k, the Born series

$$g(\boldsymbol{r}, \boldsymbol{r}') = \sum_{n=0}^{\infty} g_{2n}(\boldsymbol{r}, \boldsymbol{r}') + \sum_{n=0}^{\infty} g_{2n+1}(\boldsymbol{r}, \boldsymbol{r}') \tag{3.8}$$

converges absolutely and uniformly and that

$$\|g\| \leqslant \sum_n (\|g_1\| \, \|V\|)^n + \|g_1\| \sum_n (\|g_1\| \, \|V\|)^n,$$

whence

$$\|g\| \leqslant (1 + \|g_1\|)(1 - \|g_1\| \, \|V\|)^{-1}. \tag{3.9}$$

It is now a simple matter to verify that $g(\boldsymbol{r}, \boldsymbol{r}')$ and hence $G(\boldsymbol{r}, \boldsymbol{r}')$ defined by the Born series obey the relevant integral equations.

We remark that if k is allowed to represent a complex variable, $k = \varkappa + i\eta$ with $\eta \geqslant 0$, then (3.2) remains valid and so does (3.1) even if the limit is taken as $|k| \to \infty$. Thus the Born series for the Green's function converges in the closed upper half k plane when $|k|$ is large enough, and then we have

$$|G(\boldsymbol{r}, \boldsymbol{r}')| \leqslant \frac{\exp[-\eta |\boldsymbol{r} - \boldsymbol{r}'|]}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \times \text{constant}. \tag{3.10}$$

For the present, we restrict our study to real, positive values of k.

The convergence of the Born series for the wave function and for the scattering amplitude can be established, for sufficiently large momentum, by the

same methods. We first express the n-th order wave function in the form

$$\psi_{n+1}(\boldsymbol{r}) = \int G_n(\boldsymbol{r}, \boldsymbol{s}) V(\boldsymbol{s}) \psi_0(\boldsymbol{s}) \, \mathrm{d}\boldsymbol{s} , \tag{3.11}$$

from which we infer that

$$|r\psi_{n+1}(\boldsymbol{r})| \leqslant \|g_n\| \int \frac{r}{|\boldsymbol{r} - \boldsymbol{s}|} |V(\boldsymbol{s}) \psi_0(\boldsymbol{s})| \, \mathrm{d}\boldsymbol{s} . \tag{3.12}$$

Since $V(\boldsymbol{s})$ has been supposed to vanish for sufficiently large $|\boldsymbol{s}|$, the integral in (3.12) is both finite and bounded as a function of $\boldsymbol{r}$. Thus (3.12) shows, when use is made of the convergence of $\sum_n \|g_n\|$, that at high energies (2.6) converges uniformly and absolutely. Moreover, the wave function, apart from the initial term ψ_0, goes to zero for large r at least as fast as $(1/r)$, so that a scattering amplitude is definable. In fact, it is evident that the wave function defined by the convergent Born series satisfies the integral equation (2.1) and that for large r,

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \frac{\exp[ikr]}{r} f(\theta, \varphi) , \tag{3.13}$$

where

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} f_n(\theta, \varphi) , \tag{3.14}$$

with

$$f_n(\theta, \varphi) = \operatorname*{Lim}_{r \to \infty} r \exp[-ikr] \int G_n(\boldsymbol{r}, \boldsymbol{s}) V(\boldsymbol{s}) \psi_0(\boldsymbol{s}) \, \mathrm{d}\boldsymbol{s} \tag{3.15}$$

$$= -(4\pi)^{-1} \int \exp[-i\boldsymbol{k}' \cdot \boldsymbol{s}] V(\boldsymbol{s}) \psi_n(\boldsymbol{s}) \, \mathrm{d}\boldsymbol{s} . \tag{3.16}$$

Here, of course, $|\boldsymbol{k}'| = k$, and θ, φ are the polar angles of $\boldsymbol{k}'$ with respect to $\boldsymbol{k}$. The convergence for large k of the series (3.14) for the scattering amplitude is assured by (3.15) which implies

$$|f_n(\theta, \varphi)| \leqslant \|g_n\| \operatorname*{Max}_{r} \int \frac{r}{|\boldsymbol{r} - \boldsymbol{s}|} |V(\boldsymbol{s}) \mathrm{d}\boldsymbol{s} . \tag{3.17}$$

The discussion in the first portion of this section has already shown that $\|g_n\| \to 0$ as $k \to \infty$ for $n \geqslant 1$. Thus, we may state that as $k \to \infty$,

$$G(\boldsymbol{r}, \boldsymbol{r}') \to G_0(\boldsymbol{r}, \boldsymbol{r}') \tag{3.18}$$

or, more precisely,

$$\operatorname*{Lim}_{k\to\infty} \|g - g_0\| = 0 .$$

If k is increased while the direction of scattering is fixed, each term of (3.14) will usually go to zero. However, we may consider a limiting process in which the scattering angles are adjusted so that the incident energy goes to infinity while the momentum transfer is fixed. Then the Born approximation to the scattering amplitude, $f_0(\theta, \varphi)$, is constant whereas, by (3.17), $f_n(\theta, \varphi) \to 0$ for $n \geqslant 1$. Hence, for this process,

$$\operatorname*{Lim}_{k\to\infty} f(\theta, \varphi) = f_0(\theta, \varphi) . \tag{3.19}$$

4. – More general potentials.

The previous arguments are valid for any potential of finite strength for which (3.1) holds and for which the left side of (3.12) is bounded (so that a scattering amplitude is definable). Sufficient conditions more directly associated with the potential may also be adopted. Suppose that, for a given potential V, the function $I_V(\boldsymbol{r})$ defined in (2.15) satisfies the three requirements

$$I_V(\boldsymbol{r}) < \infty \text{ for all } r , \tag{4.1a}$$

$$I_V(r) \text{ is continuous in } r , \tag{4.1b}$$

$$I_V(r) \sim O(1/r) \text{ as } r \to \infty . \tag{4.1c}$$

It follows that $\|V\| < \infty$, and further, that V can be approximated by a simple potential in the sense that given $\varepsilon > 0$, a simple potetnial U can be constructed so that

$$\int \frac{|V(\boldsymbol{s}) - U(\boldsymbol{s})|}{|\boldsymbol{r} - \boldsymbol{s}|} \, \mathrm{d}\boldsymbol{s} \leqslant \varepsilon .$$

We forego a proof of this statement here. The proof would be tedious but is, we assert, only an exercise in epsilonics (4).

If $g_1(\boldsymbol{r}, \boldsymbol{r}')$ and $\bar{g}_1(\boldsymbol{r}, \boldsymbol{r}')$ denote the first order g functions for V and U respectively, we find, using (3.2), that

$$|g_1(r, r') - \bar{g}_1(r, r')| \leqslant \int |D_{r,r'}(\boldsymbol{s})| \, |V(s) - U(s)| \, \mathrm{d}s \leqslant \varepsilon/2\pi .$$

(4) See, for example, the methods of E. C. TITCHMARCH: *Theory of functions*, Chap. X and Sect. 12.2 (Oxford, 1939).

Hence $\lim_{k\to\infty}\|g_1\|$ is less than any positive number and must be zero. Note, also, that the right, and hence the left, side of (3.12) is bounded for such a potential.

We conclude that (4.1*a*), (4.1*b*) and (4.1*c*) may be offered as sufficient conditions on a potential to insure that the non-relativistic Born series we have discussed converge at high energies. Moreover, in the high energy limit, only the leading term of each series contributes.

Appendix

Let a function $f_S(\boldsymbol{s})$ be defined by

$$f_S(\boldsymbol{s}) = 1 \quad \text{if} \quad |\boldsymbol{s}| < S\,,$$

$$f_S(\boldsymbol{s}) = 0 \quad \text{if} \quad |\boldsymbol{s}| \geqslant S\,.$$

When $V(\boldsymbol{s})$ is a simple potential, constants S, M_1 and M_2 can be found so that for all $\boldsymbol{s}$,

$$|V(\boldsymbol{s})| \leqslant M_1 f_S(\boldsymbol{s})\,, \tag{A.1}$$

$$|\nabla V(\boldsymbol{s})| \leqslant M_2 f_S(\boldsymbol{s})\,. \tag{A.2}$$

Our objective here is to prove (3.1) of the text for simple potentials. We must show that given $\varepsilon > 0$, we can find a k_0 such that if $k \geqslant k_0$, then $|g_1(\boldsymbol{r}, \boldsymbol{r}')| < \varepsilon$ and further, that the choice of k_0 is independent of $\boldsymbol{r}$ and $\boldsymbol{r}'$. We shall put $\frac{1}{2}|\boldsymbol{r} - \boldsymbol{r}'| = \Delta$. Because the integral for $g_1(\boldsymbol{r}, \boldsymbol{r}')$ is absolutely convergent, we may express it in spheroidal co-ordinates defined by [5]

$$\left\{\begin{array}{ll} \xi = \frac{1}{2}|\boldsymbol{r} - \boldsymbol{s}| + \frac{1}{2}|\boldsymbol{r}' - \boldsymbol{s}|\,, & \Delta \leqslant \xi < \infty\,, \\ \eta = (\frac{1}{2}|\boldsymbol{r} - \boldsymbol{s}| - \frac{1}{2}|\boldsymbol{r}' - \boldsymbol{s}|)\Delta^{-1}\,, & -1 \leqslant \eta \leqslant 1\,, \\ \varphi = \text{azimuth about } (\boldsymbol{r} - \boldsymbol{r}')\,, & 0 \leqslant \varphi \leqslant 2\pi\,. \\ \mathrm{d}\boldsymbol{s} = |\boldsymbol{r} - \boldsymbol{s}|\,|\boldsymbol{r}' - \boldsymbol{s}|\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\varphi\,. & \end{array}\right. \tag{A.3}$$

Then

$$g_1(\boldsymbol{r}, \boldsymbol{r}') = \int D_{r,r'}(\boldsymbol{s}) V(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = \frac{1}{2\pi}\Delta \int \exp[2ik\xi] V(\xi, \eta, \varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\varphi\,. \tag{A.4}$$

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[5] See, for example, P. M. Morse and H. Feshbach: *Methods of Theoretical Physics*, (New York, 1953), p. 661.

Integrating with respect to ξ by parts, we obtain

$$g_1(r, r') = \tfrac{1}{4} i\pi^{-1}[\alpha_k(\boldsymbol{r}, \boldsymbol{r}') + \beta_k(\boldsymbol{r}, \boldsymbol{r}')] , \tag{A.5}$$

where

$$\alpha_k(r, r') = k^{-1}\Delta \exp[2ik\Delta]\int_r^{r'} V(\Delta, \eta, \varphi)\, \mathrm{d}\eta\, \mathrm{d}\varphi = 2\pi k^{-1} \exp[2ik\Delta]\int V(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s} \tag{A.6}$$

and

$$\beta_k(r, r') = k^{-1}\Delta\int \exp[2ik\xi]\, (\partial V/\partial\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta \mathrm{d}\varphi = \tag{A.7}$$

$$= k^{-1}\Delta\int \exp[2ik\xi] \nabla V \cdot \mathrm{d}\boldsymbol{s}_\xi\, \mathrm{d}\eta\, \mathrm{d}\varphi .$$

The path of integration in (A.6) is a straight line from $\boldsymbol{r}$ to $\boldsymbol{r}'$. And in (A.7), $\mathrm{d}\boldsymbol{s}_\xi$ is a differential of arc length in the direction of increasing ξ. Thus, employing (A.1) and (A.2), we find for (A.6) and (A.7) the bounds

$$|\alpha_k(\boldsymbol{r}, \boldsymbol{r}')| < 4\pi S M_1 k^{-1} , \tag{A.8}$$

$$|\beta_k(r, r')| < M_2 k^{-1}\Delta\int f_s(\boldsymbol{s})\, \mathrm{d}s_\xi\, \mathrm{d}\eta\, \mathrm{d}\varphi . \tag{A.9}$$

The s_ξ and η integrations in (A.9) are restricted by the condition $|\boldsymbol{s}| < S$. Select a number R so large that if both $r \geqslant R$, $r' \geqslant R$, then $g_1(\boldsymbol{r}, \boldsymbol{r}')$ as given by (A.3) is less than ε in magnitude for all k. We distinguish two cases: either $r \geqslant R$ and $r' \geqslant R$, or at least one of the variables, say r', is less than R.

In the first case, the introduction of spheroidal co-ordinates was unnecessary and no further discussion is required. In the second case, the integral in (A.9) can be estimated as follows: We first use

$$2(\Delta - \Delta\eta) = |\boldsymbol{r} - \boldsymbol{r}'| + |\boldsymbol{r}' - \boldsymbol{s}| - |\boldsymbol{r} - \boldsymbol{s}| \leqslant 2|\boldsymbol{r}' - \boldsymbol{s}| \leqslant 2r' + 2s$$

to infer

$$\Delta\eta \geqslant \Delta - (r' + s) .$$

Hence, in (A.9), the range of η is effectively limited by the inequality

$$1 \geqslant \eta \geqslant 1 - \Delta^{-1}(R + S) . \tag{A.10}$$

As ξ varies from Δ to ∞ for fixed η and φ, the vector $\boldsymbol{s}_\xi$ traces out a hyperbolic path beginning somewhere on the straight line segment running from $\boldsymbol{r}$ to $\boldsymbol{r}'$. The total length of this arc which is intercepted by the sphere $s \leqslant S$ depends on the positions $\boldsymbol{r}$ and $\boldsymbol{r}'$, but in any case, never exceeds a circumference $2\pi S$. Hence, in view of (A.10),

$$\int f_s(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}_\xi\, \mathrm{d}\eta\, \mathrm{d}\varphi \leqslant (2\pi)\Delta^{-1}(R + S)(2\pi S) ,$$

whence

$$|\beta_k(\boldsymbol{r}, \boldsymbol{r}')| < 4\pi^2 S(R+S) M_2 k^{-1} . \tag{A.11}$$

Inspection of (A.5), (A.8) and (A.11) reveals that the desired k_0 described above does indeed exist.

RIASSUNTO (*)

Lo scattering potenziale di una particella non relativistica si studia servendosi della serie di Born per la funzione di Green, la funzione d'onda e l'ampiezza di scattering. Si prova che per un'ampia classe di potenziali queste serie convergono per energie incidenti sufficientemente elevate. Si dimostra anche la validità dell'approssimazione di Born nel campo delle alte energie.

(*) *Traduzione a cura della Redazione*

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Further Experiments on Liquid Indium and Tin Self-Diffusion.

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Summary. — Further experiments on the self-diffusion coefficient D of liquid Indium and Tin give evidence of an effect of the diameter size of the capillaries used. This effect is important for Indium and is almost negligible for Tin. The values of D measured in the capillaries of the largest size (1.60 mm diameter) are taken to be the correct values for the self-diffusion coefficient. This is because using capillaries with diameter > 1.6 mm the values of D were constant in the experimental error. The dependence of D from the absolute temperature is of Arrhenius type: for Indium:

$$D = (33.4 \pm 2.1)10^{-5} \exp\left[\frac{2554 \pm 79}{RT}\right] \text{cm}^2/\text{s} ,$$

for Tin:

$$D = (32.4 \pm 1.2)10^{-5} \exp\left[\frac{-2768 \pm 80}{RT}\right] \text{cm}^2/\text{s} ,$$

1. – Introduction.

In a paper published a short time ago, A. Lodding [1] gave some data of the self-diffusion coefficient of liquid Indium at different temperatures. These data do not agree with the previous series of results published by Careri-Paoletti [2]. The two values of activation energy are not very different, but

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(1) A. Lodding: *Zeit. f. Naturfors.*, **2a**, 3, 200, 203 (1956).

(2) G. Careri and A. Paoletti: *Nuovo Cimento*, **2**, 574 (1955).

in plotting the log D versus $1/T$, Lodding's data lie on a line which is lower.

The experimental method used by LODDING is the capillary method, already used by HOFFMANN for liquid mercury [3]. The capillaries were of pyrex and quartz with a diameter of 0.55 mm. LODDING attributes the disagreement between the two series of data to some type of convection which he says might be present during CARERI and PAOLETTI's diffusion experiments. He believes that the convection is due to the large seize of the capillaries (diameter 1.6 mm).

This criticism created the necessity to re-evaluate the Careri-Paoletti method. A new series of experimental data was produced by varying the material of the capillaries (mullite, pyrex) and the diameters of the capillaries.

We found that the results obtained were dependent on the diameter and the material of the capillary used. This type of effect was never considered before by any author and its origin is not clear. In the following we shall show that this effect is peculiar of Indium.

The new data of the indium and Tin self-diffusion coefficient experiments do not substantially change the conclusion of the previous interpretation of the diffusion process by the microscopic point of view [2].

2. – Critical evaluation of our previous results.

The details of the method employed to detect the diffusion coefficient from the penetration rate of the active isotope in the inactive one, were given in (2). With the geometry used, one can calculate the self-diffusion coefficient by the following solution of the Fick equation:

$$D = \left(\frac{x}{y}\right)^2 \frac{1}{4t}. \tag{1}$$

here y is given by the equation

$$C(x) = \tfrac{1}{2}\,(1 - \operatorname{erf} y)\,, \tag{2}$$

where $C(x)$ is the concentration of diffusing material and x is the distance from the point where $C = \frac{1}{2}$.

In the previous experiments when calculating D from a given penetration curve only the experimental points lying in the regions $C = 15\,\% \div 35\,\%$ and $C = 65\,\% \div 85\,\%$ were considered.

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[3] R. E. HOFFMAN: *Journ. Chem. Phys.*, **20**, 1567 (1958).

In evaluating the error involved in each point of the penetration curve we found that the concentration interval where the error over D is not large is the interval $3\% \div 20\%$. The points lying in the regions $20\% \div 30\%$ and $70\% \div 90\%$ give values of D with a large experimental error.

A further calculation with this new criterion gives for the previous runs the results listed in Table I.

TABLE I. – *Self diffusion in liquid indium (new calculations of previous runs).*

T (°C)	255	285	451	451	635	492
$D \cdot 10^5$ cm^2/s	3.85 ± 0.21	5.46 ± 0.16	7.72 ± 0.12	7.60 ± 0.19	10.69 ± 0.34	8.84 ± 0.19

The old data corrected in this way are still higher than Lodding's data. On plotting the $\log D$ versus $1/T$, the results form a straight line which is parallel and higher than Lodding's line.

Therefore this is due to a systematic error in the calculation of D which makes D vary without influencing the slope of the straight line in the plot of $\log D$ versus $1/T$.

This difference between our data and Lodding's data can be attributed to a factor which multiplies the numerical value of D. It is clear from (1) that a systematic error (by a constant factor) may be encountered only in the variable X, which is the abscissa of the point having a concentration $C(x)$. The values of X, were measured directly on the solid rod (*) in the first method.

Thus we did not take into consideration that during solidification the rod not only shortens, but also contracts radially. Therefore the area of the cross-section of the liquid rod during the diffusion was larger than that of the solid rod. Therefore the expression used in calculating the abscissa is

$$X_{\mathrm{liq}} = X_{\mathrm{sol}} \frac{d_{\mathrm{sol}}}{d_{\mathrm{liq}}} \frac{\sigma_{\mathrm{sol}}}{\sigma_{\mathrm{liq}}} .$$

where σ_{liq} is the area of the cross-section of the rod. (σ_{sol} is known from the mass of the piece of length X_{sol}).

In order to have the true value of σ_{liq} (which is equal to the area of the section of the capillary), one must calibrate the mullite capillaries with mercury.

(*)In order to correct the effect of the specific volume change between the liquid and the solid phase, the expression was used

$$X_{\mathrm{liq}} = X_{\mathrm{sol}} \frac{d_{\mathrm{sol}}}{d_{\mathrm{liq}}} ,$$

(here d is the density of the sample).

Obviously the abscissae, so calculated are lower than the abscissae obtained before and consequently the D values are also lower. Calibrating the mullite capillaries we found that the diameter was not of the same value for every capillary, but varied from one to the other. Therefore we could not correct the first series of results, because in Careri-Paoletti's technique after each diffusion run the capillaries are broken.

3. – New experimental results.

Because of our inability to correct our first series of results, it was then necessary to have some new data. Two new series of runs using capillaries

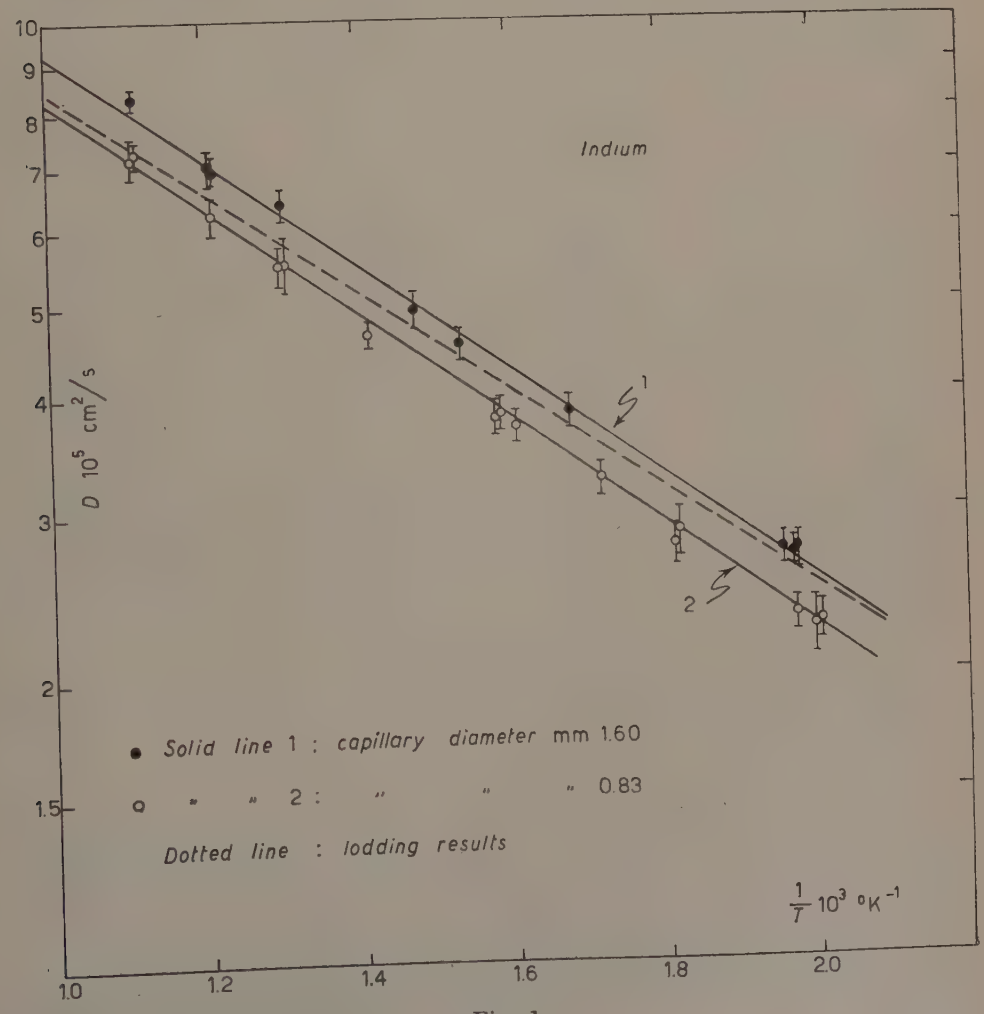

Fig. 1.

of 1.60 mm and 0.83 mm diameter, were done. The results are shown in Table II and plotted in Fig. 1.

TABLE II. – *Self-diffusion in liquid Indium*

run	T (°C)	$10^3/T$ (°K^{-1})	time	$D(10^5$ cm^2/s)
		$\Phi = 1.60$ mm		
81*a*	554	1.210	$2^h\ 57^{min}\ 30^s$	7.01 ± 0.30
81*b*	554	1.210	$2^h\ 57^{min}\ 30^s$	6.98 ± 0.25
82*a*	497	1.299	$3^h\ 09^{min}\ 30^s$	6.41 ± 0.25
85*b*	628	1.110	$3^h\ 05^{min}\ 30^s$	8.30 ± 0.20
86	407	1.471	$5^h\ 45^{min}\ 00^s$	4.92 ± 0.20
92*a*	234	1.972	$6^h\ 06^{min}\ 30^s$	2.68 ± 0.09
92*b*	234	1.972	$6^h\ 06^{min}\ 30^s$	2.70 ± 0.12
		$\Phi = 0.83$ mm		
71	310	1.715	$6^h\ 04^{min}\ 00^s$	9.24 ± 0.13
72	437	1.410	$6^h\ 26^{min}\ 00^s$	4.64 ± 0.14
73*c*	359	1.580	$3^h\ 43^{min}\ 00^s$	3.78 ± 0.15
73*d*	359	1.580	$3^h\ 43^{min}\ 00^s$	3.80 ± 0.16
81*d*	554	1.210	$2^h\ 57^{min}\ 30^s$	6.23 ± 0.30
82*b*	497	1.299	$3^h\ 09^{min}\ 30^s$	5.51 ± 0.28
82*c*	497	1.299	$3^h\ 09^{min}\ 30^s$	5.52 ± 0.38
85*c*	628	1.110	$3^h\ 05^{min}\ 30^s$	7.18 ± 0.35
85*d*	628	1.110	$3^h\ 05^{min}\ 30^s$	7.26 ± 0.22
92*c*	234	1.972	$6^h\ 06^{min}\ 30^s$	2.31 ± 0.09
123*a*	278	1.815	$4^h\ 28^{min}\ 30^s$	2.84 ± 0.17
123*b*	278	1.815	$4^h\ 28^{min}\ 30^s$	2.74 ± 0.14
104*a*	227	2.000	$6^h\ 23^{min}\ 00^s$	2.25 ± 0.17
104*b*	227	2.000	$6^h\ 23^{min}\ 00^s$	2.27 ± 0.11
		run in pyrex capillary		
04	227	2.000	$6^h\ 23^{min}\ 00^s$	2.52 ± 0.12

In Table III and in Fig. 2 are shown the results of the self-diffusion of tin, which we had to rerun because of the impossibility of correcting the previous results. With tin we also made two series of measurements with capillaries of different diameters.

Upon plotting the Indium results as $\log D$ versus $1/T$ two straigth lines were formed, corresponding to the two runs of the capillaries of 1.60 mm and 0.83 mm.

TABLE III. – *Self-diffusion in liquid Tin.*

run	T (°C)	$10^3/T$ °K^{-1}	time	$D(10^5$ cm^2/s)
		$\Phi = 1.60$ mm		
108*a*	414	1.455	$4^h\ 06^{min}\ 00^s$	4.52 ± 0.23
108*b*	414	1.455	$4^h\ 06^{min}\ 00^s$	4.49 ± 0.19
109*a*	526	1.251	$3^h\ 26^{min}\ 00^s$	5.76 ± 0.28
109*b*	526	1.251	$3^h\ 26^{min}\ 00^s$	5.57 ± 0.19
111*a*	353	1.598	$7^h\ 14^{min}\ 30^s$	3.29 ± 0.15
111*b*	353	1.598	$7^h\ 14^{min}\ 30^s$	3.32 ± 0.16
112*a*	302	1.738	$6^h\ 40^{min}\ 00^s$	2.84 ± 0.12
112*b*	302	1.738	$6^h\ 40^{min}\ 00^s$	2.95 ± 0.12
113*a*	683	1.045	$3^h\ 08^{min}\ 30^s$	7.93 ± 0.45
		$\Phi = 0.83$ mm		
108*c*	414	1.455	$4^h\ 06^{min}\ 00^s$	4.26 ± 0.17
109*c*	526	1.251	$3^h\ 26^{min}\ 00^s$	5.53 ± 0.35
111*c*	353	1.598	$7^h\ 14^{min}\ 30^s$	3.48 ± 0.17
111*d*	353	1.598	$7^h\ 14^{min}\ 30^s$	3.33 ± 0.18
112*c*	302	1.738	$6^h\ 40^{min}\ 00^s$	2.82 ± 0.14
113*c*	683	1.045	$3^h\ 08^{min}\ 30^s$	7.54 ± 0.39
113*d*	683	1.045	$3^h\ 08^{min}\ 30^s$	7.32 ± 0.26
115*c*	291	1.672	$6^h\ 16^{min}\ 00^s$	2.73 ± 0.14
115*d*	291	1.772	$6^h\ 16^{min}\ 00^s$	2.88 ± 0.14
116*c*	267	1.852	$7^h\ 40^{min}\ 00^s$	2.40 ± 0.12

These lines are parallel within the experimental error and approach the parallel of the Lodding's straight line, which is found to be intermediate between the two. This is clear from the least squares equations, which represent our data:

$$D = (33.4 \pm 2.1) \cdot 10^{-5} \exp\left[-\frac{2554 \pm 79}{RT}\right] \text{cm}^2/\text{s} \qquad \Phi = 1.60 \text{ mm}\,.$$

$$D = (30.2 \pm 2.0) \cdot 10^{-5} \exp\left[-\frac{2580 \pm 83}{RT}\right] \text{cm}^2/\text{s} \qquad \Phi = 0.83 \text{ mm}\,.$$

and Lodding's data

$$D = (28.9 \pm 2.5) \cdot 10^{-5} \exp\left[-\frac{2430 \pm 50}{RT}\right] \text{cm}^2/\text{s}\,.$$

From Fig. 2, one can see that a diameter effect is not appreciable in tin.

As a matter of fact, all experimental data are well represented by the equation

$$D = (32.4 \pm 1.2) \cdot 10^{-5} \exp\left[-\frac{2768 \pm 80}{RT}\right] \text{cm}^2/\text{s} .$$

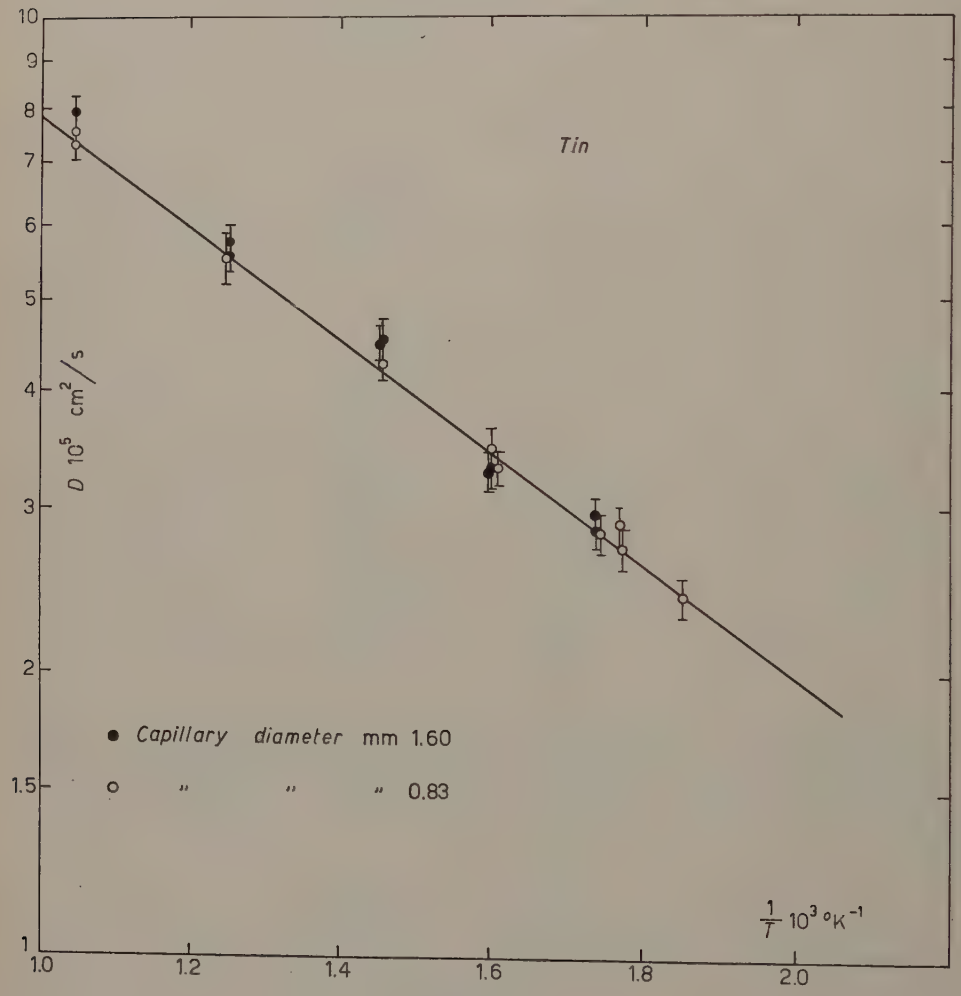

Fig. 2.

4. – Discussion.

The above results can not be explained by some sort of convection either in our measurements or in Lodding ones. Such an explanation could have been consistent with our old results, in spite of the reproducibility with different times of diffusion, because it is easier to have convection in large tubes

than in thin ones. But now it is clear that our old measurements were large only due to the systematic error in the abscissa. And in fact the correction factors which would be necessary to bring the old data to coincide with the new ones, are of the same order of the factors effectively used in the new experimental series of runs.

Another reason for not speaking in terms of convection is that we made a check of Lodding's results with our technique and did find a very good agreement, within the experimental error (see run 104). We did not use a capillary of pyrex with exactly the same diameter as LODDING used, but a diameter of $\simeq 1.0$ mm. So we would have expected, if the convection was the cause of the difference between the results, a larger D than Lodding found. This didn't happen. So we must attribute the difference existing between the results of:

a) the runs in mullite capillaries with $\Phi = 1.60$ mm,

b) the runs in mullite capillaries with $\Phi = 0.83$,

c) the runs in pyrex capillaries,

to something which is not convection if by « convection » we mean some long-range phenomena of microscopic mixing due to causes like negative thermal gradients, vibrations of the apparatus and other accidental causes.

5. – The « wall effect ».

This unknown effect can be a « wall effect ». We mean by that that the molecules which are near the wall of the capillary do not participate in the diffusion in the same way as the molecules of the core.

One can understand at once that if this « wall effect » exists, it will give for the diffusion in tubes values of D which are lower than the true ones and that the effect is larger the thinner are the tubes in which the diffusion takes place. This is because in the two methods of analysis we are considering (our method and Lodding's one), the quantity that is directly measured is the variation of the concentration of the whole mass present in the capillary. If all the material in the tube is diffusing this leads to the correct value of D, but if one supposes that only the core molecules of the rod are diffusing, while the molecules which stay near the walls are not, one must conclude that the true value of D must be larger than the measured one.

Obviously one can't make the crude approximation of considering a variation of concentration with the radius of the tube as a step function ($c = 0$ for $r_0 \leqslant r \leqslant R$; $c = c_0$ for $0 \leqslant r \leqslant r_0$; where R is the radius of the capillary, r_0 is the radius of the core where the molecules do diffuse); because this ap-

proximation does not preserve the features of the function

$$c(x) = \tfrac{1}{2}(1 - \operatorname{erf} x/\sqrt{4Dt}) .$$

The proper procedure would be to obtain the solution of the Fick equation not in the unidimensional case but in cylindrical co-ordinates with independent variables X, r and t, in the case of a cylinder of a material with a diffusion co-efficient D_1 enclosed by a mantle of thickness η of a material with a diffusion coefficient D_2 and $D_2 < D_1$ and perhaps $D_2 = 0$, and supposing that at the separation surface between the two materials there could be a radial diffusion caused by the concentration gradient which exists because of the axial diffusion. Until now this has not been done.

All the same we can think that at the end of the diffusion the variation of c is larger in the core of the rod than in the external part.

The thickness of this cylindrical zone can be thought of as being dependent not on the diameter of the tube but only on the material of which the tube is made and on the diffusing metal. Then the D value will be lowered as the capillary becomes thinner, but if the diameters are quite large the effect will be practically negligible. If in some way, one could remove the external part from a metallic rod before its analysis, to analyze only the core, (the region in which the concentration variations are larger), one would obtain a higher D value. This has been done to test this « wall effect » hypothesis. By immersion of the metallic rod in nitric acid $\simeq 100$ μm of the external part was removed. The results are shown in Table IV together with the results of some diffusion runs in mullite capillaries of 1.00 and 2.00 mm diameters. The results, as one can see, agree with our hypothesis, since the runs in nitric acid and in capillaries with $\Phi = 2.00$ mm give values of D which are a little larger than the values in capillaries with $\Phi = 1.60$ mm. The runs in tubes with

TABLE IV. – *Self diffusion in liquid Indium.*

run	Φ (mm)	T (°C)	$10^3/T$ (°K^{-1})	time	$D(10^5$ cm^2/s)
80*b*	1.60 mm	381	1530	$4^h\ 10^{min}\ 00^s$	4.52 ± 0.17
80*a*	1.60 mm in	»	»	»	4.64 ± 0.25
80*d*	HNO_3	»	»	»	4.87 ± 0.25
83*b*	1.60 mm	324	1675	$3^h\ 37^{min}\ 00^s$	3.82 ± 0.15
83*c*	1.60 mm in	»	»	»	3.96 ± 0.20
83*d*	HNO_3	»	»	»	4.04 ± 0.20
83*e*		»	»	»	4.04 ± 0.20
91*b*	1.60	239	1955	$5^h\ 27^{min}\ 00^s$	2.71 ± 0.11
91*f*	1.00	»	»	»	2.42 ± 0.10
10_{2b}	0.83	350	1675	$3^h\ 20^{min}\ 30^s$	3.69 ± 0.10
10_{2a}	2.00	»	1605	»	4.39 ± 0.15

$\Phi = 1.00$ mm give values of D which are intermediate between the two series with $\Phi = 0.83$ and 1.60 mm.

We also have shown in Table V some preliminary results obtained using an alloy 1% Lead — 99% Indium (radioactive tracer RaD). This alloy shows a « wall effect » similar to that of Indium but which a smaller value.

TABLE V. – *Self diffusion in an alloy* 99% *Indium* 1% *Lead.*

run	Φ (mm)	T (°C)	$10^3/T\,°K^{-1}$	time	$D(10^5\ cm^2/s)$
95*a*	1.60	487	1315	$3^h\ 17^{min}\ 00^s$	4.75 ± 0.19
95*b*	»	»	»	»	4.75 ± 0.19
95*c*	0.83	»	»	»	4.51 ± 0.18
95*d*	0.83	»	»	»	4.55 ± 0.18
106*a*	1.60	388	1512	$4^h\ 21^{min}\ 00^s$	3.91 ± 0.16
106*b*	1.60	»	»	»	3.79 ± 0.15
106*c*	0.83	»	»	»	3.66 ± 0.18
106*d*	0.83	»	»	»	3.60 ± 0.18

6. – Concluding remarks.

I. – Let us briefly summarize all the above matter:

a) The measurements of the self-diffusion in Indium give different results on varying the diameters of the mullite tubes. The tubes used had diameters of 0.83 mm, 1.00 mm; 1.60 mm and 2.00 mm. The numerical values of D are larger the larger the diameters.

b) Removing with nitric acid the external part of the rods after diffusion had taken place, the numerical values of D were a little higher.

c) Changing the material of which the tubes are made (mullite, pyrex) the results were different and the difference could be explained supposing that the molecules near the walls of the tubes did not diffuse.

d) Changing the diffusing metal one could observe:

1) no detectable difference between the runs in tubes with different diameters (Tin);

2) a smaller difference than in indium: (alloy 1% Lead — 99% Indium).

From these facts one concludes that it is not a question of convection because we can't see any reason for the presence of different type of convection in Indium, in Tin and the alloys.

So we have supposed the existence of a « wall effect » which is peculiar to indium (the presence of the effect in the alloys we can attribute to the presence of a great deal of indium compared with the other metal). However, analyzing the surface and structural properties of indium and tin and making a comparison between them we found difficult to explain our wall effect. We must remember that E. DRICKAMER [4] also found with indium a strange surface effect, which can be related to our « wall effect » because also in the Drickamer measurements the diffusion coefficient of the surface zone was found to be lower than the D value in the mass. Also DRICKAMER did not succeed in explaining this effect and he assumes that the correct D value is the higher one.

Having seen that the runs in which the external part of the rod has been carried away with nitric acid and the runs in capillaries with $\Phi = 2.00$ mm give not much larger numerical values of D than those measured with capillaries of $\Phi = 1.60$ mm, we can think that the correct value of the self-diffusion coefficient of liquid indium, meaning by correct value the value in the core of the rod, is approximately given by

$$D = (33.4 \pm 2.1) \cdot 10^{-5} \exp\left[-\frac{2554 \pm 79}{T}\right] \text{cm}^2/\text{s} .$$

For the self-diffusion coefficient of tin we don't have any wall effect so that the diffusion coefficient is well represented by

$$D = (32.4 \pm 1.2) \cdot 10^{-5} \exp\left[-\frac{2768 \pm 80}{T}\right] \text{cm}^2/\text{s} .$$

II. – We want here to discuss briefly if our new results change somehow the interpretation of the mechanism of diffusion by the microscopic point of view given in a previous work (G. CARERI and A. PAOLETTI: *Nuovo Cim.*, **2**, 574 (1955)).

There it was shown that if one wanted to describe diffusion with a quasi-crystalline model one had to assume a direct exchange mechanism in order to explain the experimental facts.

The new experimental results show that instead of a slight curvature of the function $\log D = f(1/T)$, one has quite a straigth line which follows Arrhenius' equation.

In the treatment given in the previous paper, D was expressed as a function of T and of two parameters A and H_1 and the values of these parameters were determined by a method of « best fitting » the experimental points.

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[4] H. G. DRICKAMER and R. E. ECKERT: *Journ. Chem. Phys.*, **20**, 13 (1952).

Supposing for ε the value $3.6 \cdot 10^{-4}$ and the f.c.c. structure me have seen the new experimental points are well fitted with the following values for H_1 and A

	H_1	A
Indium	3630	0.071
Tin	4020	0.050

All the previous discussion may then be applied to the new results and we shall recall here only the conclusion: it seems more probable to think that a liquid metal, which can be thought of as a quasi-crystal, diffuses by a mechanism of direct exchange than by migration via vacancies, these two being the typical processes of diffusion in the crystalline state.

RIASSUNTO

Una nuova serie di misure del coefficiente di autodiffusione D dell'Indio e dello Stagno liquidi mostra l'influenza della sezione del capillare sul valore numerico di D. Questo effetto è importante per l'Indio e quasi trascurabile per lo Stagno. Si assumono come valori corretti di D i valori ottenuti con i capillari di diametro maggiore (1.6 mm) avendo visto che per diametri ancora più grandi D restava costante entro l'errore. La dipendenza di D dalla temperatura assoluta segue la legge di Arrhenius:
per l'Indio:

$$D = (33.4 \pm 2.1) \cdot 10^{-5} \exp\left[\frac{2554 \pm 79}{RT}\right] \text{cm}^2/\text{s} ,$$

per lo Stagno:

$$D = (32.4 \pm 1.2) \cdot 10^{-5} \exp\left[\frac{-2768 \pm 80}{RT}\right] \text{cm}^2/\text{s}.$$

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

Interaction at Low Energies of π^+ Mesons in Photographic Plates.

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(ricevuto il 13 Ottobre 1958)

Summary.—The interactions of π^+-mesons, in the energy range $(0 \div 60)$ MeV, in nuclear emulsions is investigated. The model which assumes that the pion absorption in nuclei occurs via a neutron proton pair is discussed.

1. - Experimental results.

A detailed investigation has been done of the interaction of π^+ mesons with the nuclei of the nuclear emulsions in the energy range $(0 \div 60)$ MeV [1].

To do this we have used a stack of Ilford G-5 photographic plates exposed to the 62 MeV π^+ meson beam of the Nevis cyclotron, and a stack of Ilford G-5 stripped emulsions exposed to the 59 Meson beam of the Chicago cyclotron. The stripped emulsion stack, with meson energies ranging from 59 down to 0 MeV, has been divided into the three smaller energy intervals: $(55 \div 35)$ MeV, $(35 \div 15)$ MeV, $(15 \div 0)$ MeV.

The correction due to the straggling and the multiple Coulomb scattering has been made by taking into account the results of the analysis of 1000 $\pi^+ \to \mu^+$ decays.

From a phenomenological standpoint the interactions observed are divided into the three group:

1) Elastic scatterings,

2) Inelastic scatterings.

3) Stars.

For the processes 1), 2) we have used only the track-scanning data, and for

[1] Preliminary results have been published: A. Minguzzi, G. Puppi and A. Ranzi: *Nuovo Cimento*, **11**, 697 (1954).

process 3) also the area-scanning data. All the scatterings have been analyzed as a function of the parameter $\Delta E/E$ (*i.e.* the fractional loss of energy of the incoming pion) because our beams are not strictly monochromatic.

The distribution of the scatterings versus $\Delta E/E$ has been done by taking into account the experimental errors. The form of the distribution corresponding to $\Delta E/E < 0$ suggests to define as elastic all the scatterings with $\Delta E/E < 25\%$.

The mean energies for the incoherent events (2, 3) and for the elastic one has been calculated by weighting the energy spectrum of the incoming pions with the energy dependence of the corresponding cross-sections. The number of events and the mean free paths for the various processes are indicated in Table I and II.

TABLE I.

Mean energy (MeV)		65 ± 7	47 ± 5	27 ± 7	12 ± 4
stars+stops	N	301	205	107	73
	λ (cm)	39.9± 3.6	65.9± 6.3	71.0± 9.8	87.2±15
inel. scatt.	N	72.8	21.4	6.5	0
	λ (cm)	133.1±22	146.5±34.9	295.0±113.0	∞
tot. incoh.	N	373.8	226.4	113.5	73
	λ (cm)	30.8± 2.7	45.7± 4.5	57.2± 15	87.2±15
$\sigma_s/\sigma_a = \lambda_a/\lambda_s$		0.299	0.450	0.241	0

TABLE II.

Mean energy (MeV)		62 ±7	43 ±5	20.5±7	8.5 ±4
el. scatt.	N	>12° 202	>6° 161	>11° 181	>11° 275
	λ (cm)	53.9±5.6	16.7±1.3	9.7±0.7	2.41±0.14

2. - Interpretation of the results in terms of the optical model.

In order to compare our results with the elementary pion-nucleon interactions, we have calculated the absorption coefficients ($k_t = 1/\lambda_t$) for pions

in nuclear matter. « k_t » is simply related to the total incoherent cross-section in the approximation of the optical model of S. FERNBACH, R. SERBER and T. B. TAYLOR (2). Because we are interested in very low energies, we introduced in this model the modification arising from the Coulomb field (3).

The results are tabulated in Table III.

TABLE III.

Mean energy (MeV)	65±7	47±5	27±7	12±4
$k_t = (1/\lambda_t)(10^{13}$ cm)	.334±.060	.200±.035	.137±.025	.049±.020
corrected σ_t/σ_a	.676	.450	.241	—
$k_a = (1/\lambda_a)(10^{13}$ cm)	.199±.031	.138±.025	.110±.025	—
$k_s = (1/\lambda_s)(10^{13}$ cm)	.135±.035	.062±.015	.027±.012	—

From the experimental ratios between inelastic scattering and true absorptions one can calculate the inelastic scattering coefficients ($k_s = 1/\lambda_s$) and the true absorption coefficients ($k_a = 1/\lambda_a$). We notice however that, when the nuclear dimension is large in respect to λ_t, one must take into account the fact that more than one meson interaction can take place in nuclear matter. This happens only in the highest energy interval.

The ratios σ_s/σ_a referred to the first interaction are corrected according to the calculation of E. CLEMENTEL and G, PUPPI (4), and the results are summarized in Table III.

In Fig. 1 and Fig. 2 are plotted k_s and k_a respectively versus the energy of the incoming pions. The meson-nucleon scattering cross-section at low energies is roughly proportional to the fourth power of the pion momentum q (5). Fig. 1 shows that also k_s is nearly proportional to q^4. This fact supports the hypothesis that the main part of the inelastic scatterings arises from pion nucleon scattering.

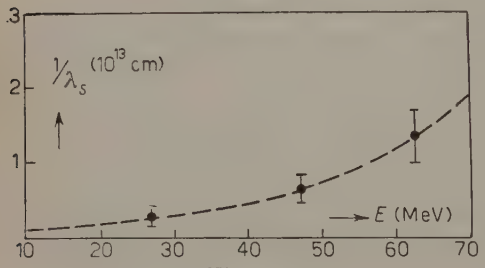

Fig. 1.

(2) S. FERNBACH, R. SERBER and T. B. TAYLOR: *Phys. Rev.*, **75**, 1352 (1949).
(3) K. M. GATHA and R. J. RIDDEL jr.: *Phys. Rev.*, **86**, 1035 (1952).
(4) E. CLEMENTEL and G. PUPPI: *Nuovo Cimento*, **10**, 197 (1953).
(5) N. C. FRANCIS and K. M. WATSON: *Amer. Journ. of Phys.*, **21**, 659 (1953).

The cross section for the process

$$\pi^+ + \mathrm{d} \rightarrow \mathrm{p} + \mathrm{p} \tag{1}$$

at low energies goes roughly as q^2/β (5) and Fig. 2 shows that also k_a has the same behaviour. This fact supports the hypothesis that the main part of the true absorption goes along the reaction channel (1).

According to (6), the probability of true absorption of pions in nuclei (W_a) is related to the cross-section (σ_D) of the elementary process (1) as

$$W_a = \Gamma \sigma_D ,$$

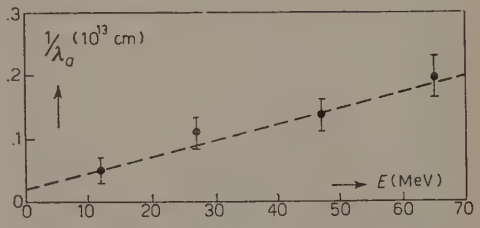

Fig. 2.

where Γ is the ratio of the probability distribution of the neutron proton pairs in heavy nuclei and in deuterium.

Γ must be independent of the meson energy and we have found $\Gamma \sim 10$ for all the energies.

3. - Fast protons from π^+ meson absorptions.

The results of the absorptions have been investigated in more detail and compared with the model which assumes that the pion absorption follows from the reaction $\pi^+ + \mathrm{d} \rightarrow \mathrm{p} + \mathrm{p}$ (1).

The maximum kinetic energy ($T_{c\max}$) of a proton pair produced in a nucleus is

$$T_{c\max} = E_\pi - E_s ,$$

where E_π is the total energy of the π-meson and E_s is the separation energy of a neutron proton pair from a given nucleus.

The probability that a proton pair possesses a kinetic energy lower than $T_{c\max}$ is calculated, according to the Fermi model of the nucleus, by observing that the number of levels of an uncorrelated neutron proton pair, per unit kinetic energy, is proportional to the square of its kinetic energy. From this follows that the choice of the value of the Fermi well depth does not significantly affect the probability distribution.

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(6) K. A. Brückner, R. Serber and K. M. Watson: *Phys. Rev.*, **84**, 258 (1951); G. Bernardini and F. Lévy: *Phys. Rev.*, **84**, 610 (1952).

We have assumed that the Ilford G-5 nuclear emulsion is composed of Ag, Br, O and C nuclei and for each of these nuclei we have calculated the corresponding $T_{c\max}$ and the neutron proton level distribution.

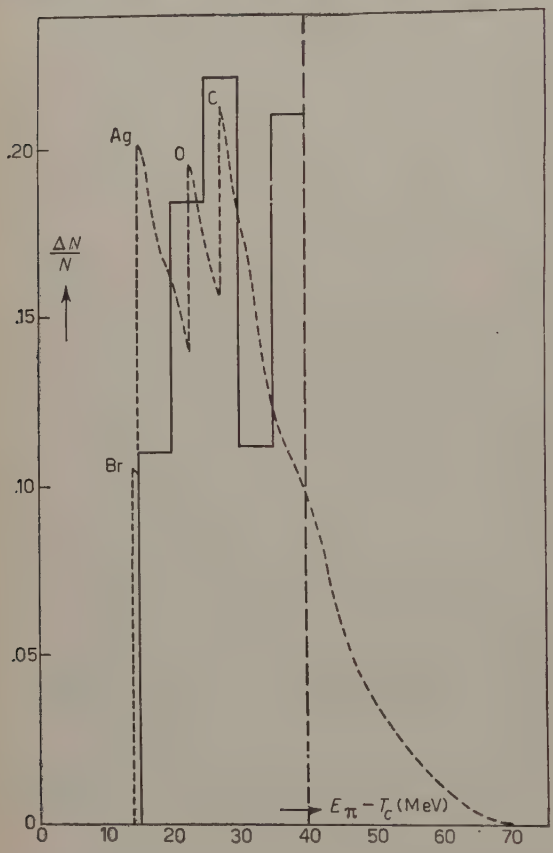

Fig. 3. – Proton pairs distribution versus $E_\pi - T_c$ in MeV (the dashed line is the theoretical and the full line is the experimental one with a cut-off $E_\pi - T_c \leqslant 40$ MeV).

We have weighted the percentage of each nucleus in Ilford G-5 over the corresponding absorption cross-sections and the result is that: 30% of the absorptions are produced on Ag, 30% on Br, 20% on O, and 20% on C. If one supposes that all the absorptions follow from the reaction (1), the expected proton pair distribution versus $E_\pi - T_c$ is plotted in Fig. 3.

The theoretical distribution shows that no proton pair can exist with an $E_\pi - T_c$ greater than 14 MeV, and we have found no pairs at all violating this condition. On the other hand the theoretical distribution suggests that a safe criterion for selecting proton pairs, which have not interacted in nuclear matter (A events), is given by

a) $\quad E_\pi - T_c \leqslant 40$ MeV.

The proton pairs chosen according to this cut-off are also plotted in Fig. 3.

We have calculated the total momentum of the neutron proton pair inside the nucleus (P_c) of the A events. Only 5% of these events has a total momentum exceeding twice the maximum momentum allowed by the Fermi sphere ($P_{c\max}$). This fact can be justified if one uses a nuclear model more realistic than a square well.

In Fig. 4 we have plotted the distribution of the momentum of the pairs with $P_c \leqslant 2P_{c\max}$ versus $P_c/P_{0\max}$, compared with the distribution of the total momentum of an uncorrelated neutron proton pair in the Fermi sphere.

The distribution of the relative angles (φ_c) between the incoming pion direction and the total momentum P_c for the A events is plotted in Fig. 5.

The figures 4 and 5 show that the absorption probability of the π^+ mesons by neutron proton pairs increases with P_c and φ_c; this result is in qualitative agreement with the energy behavior of the cross-section of the reaction (1).

If one takes into account the nuclear transparency for protons

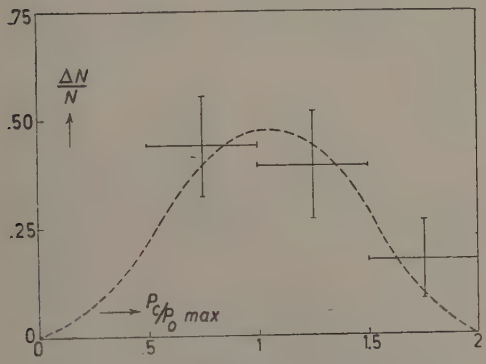

Fig. 4. – Distribution of the total momentum of the neutron proton pair inside the nucleus. The dashed line is the random distribution in the Fermi sphere. The full line is the experimental one.

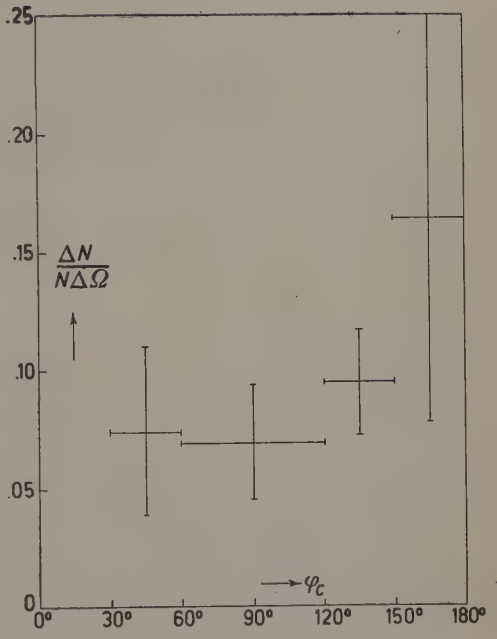

Fig. 5. – Distribution of the A events versus φ_c.

with a kinetic energy ranging from 60 to 100 MeV, one obtains that the probability for a proton pair to escape nuclear interaction ranges from 5%, to 15% of the number of the primary absorptions.

The experimental ratio between the number of A events (corrected taking into account the tails, beyond the cut-off, of the theoretical distribution of Fig. 4) and the number of the primary absorptions is of the order of 7%, in agreement with the theoretical prevision.

We have also compared the angular distribution of the protons of the A

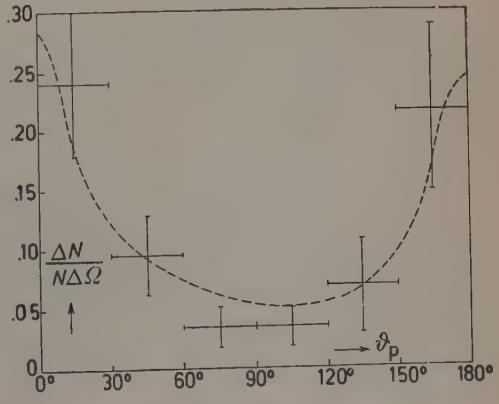

Fig. 6. – Angular distribution of the protons of the A class of events: the dashed line is the Montecarlo distribution, the full line is the experimental one.

events, in respect of the incoming pion direction, with the angular distribution of the protons outgoing from the reaction (1).

To do this we have performed a Montecarlo calculation by assuming a random distribution of a neutron proton pair in the nucleus and by taking, for the reaction (1), the experimental results of reference [7].

The Fig. 6 shows that the angular distribution of the protons of the A events and the angular distribution predicted by the Montecarlo calculation are in good agreement.

For further references we notice that the peak in the forward direction is mainly due to the fastest protons.

All the previous arguments boil down to the conclusion that the criterion *a*) is reliable for selecting the proton pairs which have not interacted.

One can investigate another class of events (B) selected in such a way that one proton of the pair escapes the nucleus without interaction. The criterion *a*) suggests that such a proton must have a kinetic energy T_p satifying

b) $$T_p \geqslant \frac{E_\pi - 40}{2} .$$

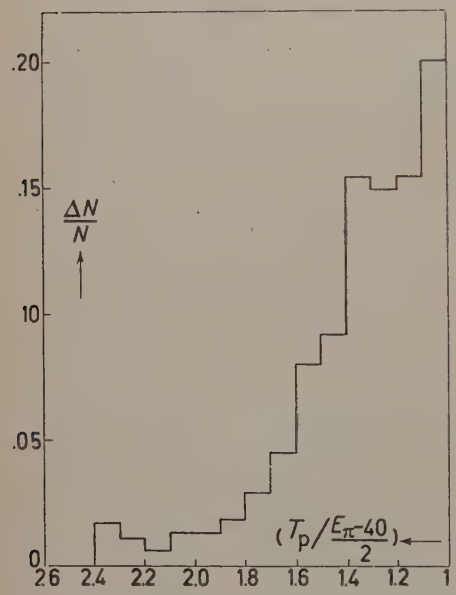

Fig. 7. – Distribution of protons of the B class versus $T_p/(E_\pi - 40)/2$.

In the class B only the fastest proton of the type A events is always included.

We have plotted the distribution of the protons of the class B versus $T_p/(E_n - 40)/2$ in Fig. 7.

The Fig. 7 shows that the largest probability occurs for a quasi-equal distribution of the released energy between the two protons of the pair.

This distribution can be useful for estimating the pion production in the nuclei by strange particles.

As we did before for the protons of the A class, if one takes into account the transparency for protons in nuclear matter one obtains that the probability that one proton of the pair escapes interaction, whatever may happen to the second proton, ranges from 16% to 29% of the number of the primary interactions. The experimental result of this ratio is 24.5%.

We have plotted in Fig. 8 the angular distribution of these protons. The comparison with the previous Montecarlo calculation shows that the experimental distribution is more pronounced in the forward direction and more

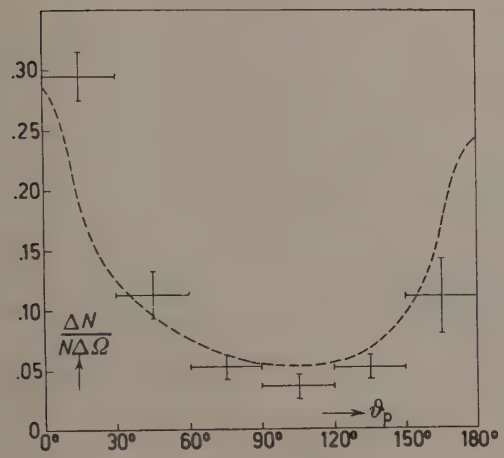

Fig. 8. – Angular distribution of the protons of the B class (the dashed line is the Montecarlo distribution, the full line is the experimental one).

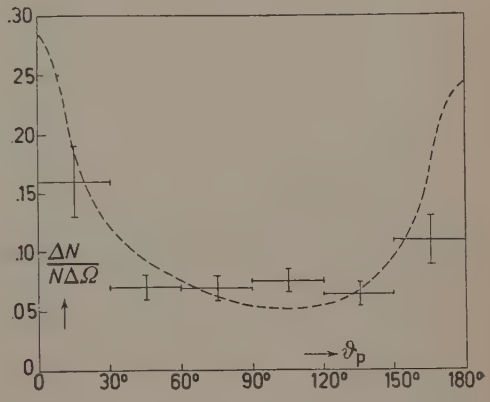

Fig. 9. – Angular distribution of the protons of all the absorptions (the dashed line is the Montecarlo distribution the full line is the experimental one).

depressed backwards. Such a deformation is to be expected because the condition *b*) emphasizes the fastest protons, and these, according to a previous remark, are peaked in the forward direction.

In Fig. 9 we have plotted the angular distribution of the protons of all the absorptions.

In this distribution the feature of the angular distribution of the protons of the reaction (1) are also recognizable although smoothed because of the successive interactions of the protons.

The conclusion is that the reaction $\pi^+ + d \to p + p$ is effectively the most important channel for the absorption of π^+ mesons, without excluding that sometimes the absorption via neutron-neutron pair or more complicated processes can take place.

* * *

We like to acknowledge the useful discussions with Prof. G. Puppi during the course of the work and in attemps to explain the results, and the encouragement of Prof. G. Bernardini.

Prof. G. Bernardini and Dr. J. Orear are gratefully thanked for supplying the nuclear plates.

We acknowledge also the contribution to this work by Miss A. MONREALE, Miss C. SOAVE, Dr. G. FOGAGNOLO and Mr. A. ZAMBONI, who scanned the plates and by Dr. A. QUARENI-VIGNUDELLI, Miss R. SARTONI and Mr. L. BRIZIO who carried out most of the numerical calculations.

RIASSUNTO

Si è studiata l'interazione in emulsione nucleare di mesoni π^+ di energia tra 0 e 60 MeV. È discusso il modello secondo cui l'assorbimento di un π^+ in un nucleo avviene tramite una coppia protone-neutrone.

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Photoproduction of Charged Pions in Deuterium.

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(ricevuto il 16 Ottobre 1958)

Summary. — An experimental analysis of the process

$$h\nu + \mathrm{D} \begin{cases} \nearrow \pi^- + \mathrm{p} + \mathrm{p} \\ \searrow \pi^+ + \mathrm{n} + \mathrm{n} \end{cases}$$

is presented. The $\sigma(-)/\sigma(+)$ ratio has been measured in the photon energy interval (170÷230) MeV and Lab. angles 45°, 75°, 105°, 150°. The results are interpreted on the base of the impulse approximation with the aim of getting information on the process $h\nu + \mathrm{n} \to \pi^- + \mathrm{p}$.

1. – Introduction.

Today the fact seems rather obvious, although still quite remarkable, that in spite of the zero-total electric charge of the neutron, the photo-pion production by the neutron is comparable and even larger than that due to the proton. It is obvious because it is known that the dominant terms of the photo-pion production by a nucleon in the range between the pion-threshold and 400 MeV are (1):

1) the *S*-wave, required by gauge-invariance;
2) the *P*-wave, mostly due to the magnetic dipole absorption; and
3) the direct photo-effect of the pion cloud.

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(×) Now at «Fermi Institute for Nuclear Studies», University of Chicago, USA.
(1) *e.g.* G. F. Chew and F. W. Low: *Phys. Rev.*, **101**, 1579 (1956).

All these are expected to be about the same both for the proton and neutron.

In fact, according to this nucleon-structure picture of the photo-pion production, the only difference which is expected between the twin reaction

(1) $$\gamma + n \to p + \pi^-$$

(2) $$\gamma + p \to n + \pi^+$$

should be found in the existence of appreciable terms due to nucleon recoil [2].

In the last few years some experiments [3] were undertaken to investigate reaction (1) making use of a deuterium target. The information regarding this reaction was derived from the assumption that, to some extent, a deuterium target is equivalent to a free neutron plus a free proton (*).

With a somewhat equivalent programme (*i.e.* with the hope of measuring the ratio $(\pi^-)/(\pi^+)=d\sigma^-/d\sigma^+$ among the differential cross-sections of reactions (1) and (2) in a fairly narrow energy range but at several angles), another experiment was again tried on the photoproduction of charged pions from deuterium.

It was a natural extension immediately following an experiment in hydrogen which has already been reported [4].

In spite of the use of the nuclear emulsion (as pion detector) an effort was made to obtain a fairly large statistic in order to reach a good accuracy.

As is well known, when doing the analysis of the data the main uncertainties were not raised by the quality of the experimental results, but from the interpretation and discussion of the three-body reaction

(3) $$\gamma + D \begin{cases} \nearrow (p + p) + \pi^- \ (a) \\ \searrow (n + n) + \pi^+ \ (b) \end{cases}$$

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[2] K. WATSON: *Phys. Rev.*, **85**, 852 (1952).

[3] I. L. LEBOW, B. T. FELD, D. H. FRISCH and L. S. OSBORNE: *Phys. Rev.*, **85**, 681 (1952); R. M. LITTAUER and D. WALKER: *Phys. Rev.*, **86**, 838 (1952); S. WHITE, M. J. JACOBSON and A. G. SCHULZ: *Phys. Rev.*, **88**, 836 (1952); J. L. JENKINS, D. LUCKEY, T. R. PALFREY and R. R. WILSON: *Phys. Rev.*, **95**, 179 (1954); M. SANDS, J. G. TEASDALE and R. L. WALKER: *Phys. Rev.*, **95**, 592 (1954); D. C. HAGERMAN, K. M. CROWE and R. M. FRIEDMAN: *Phys. Rev.*, **106**, 818 (1957).

(*) The actual amplitude of reaction (1) could be measured by the radiative scattering, *i.e.* by the reverse reaction of (1). But so far this experiment seems to be encountering quite substantial experimental difficulties for the competing reaction $\pi^- + p \to n + \pi^0$ which has a cross-section from 10 to 100 times larger.

[4] M. BENEVENTANO, G. BERNARDINI, D. CARLSON-LEE, G. STOPPINI and L. TAU: *Nuovo Cimento*, **4**, 323 (1956) — hereafter referred to as I.

when the continuous photon bremsstrahlung spectrum is taken into account. Besides the complications due to the kinematics, this analysis has to face the unsolved problem of the two-nucleon forces. In this analysis use has been made of the impulse approximation. This is practically equivalent to neglecting the nuclear forces in respect of those forces which originate the processes (3). The plausibility of this approximation will be justified on the basis of theoretical and experimental arguments.

Some preliminary accounts of this work have already been reported ([4,5]). The slight differences which may be noticed in the figures etc., between the data here presented and the others previously published, are due to improvements in the statistics which are now based on a total of more than twenty thousand clearly identified pions. The differences now mentioned have been found to be systematically between the error limits.

After publication of the reports to which we now refer, it was felt that they could have been considered an adequate and sufficient presentation of our results, because in any case they could not be extremely conclusive for the several uncertainties involved in the interpretation of the data. But our results, and a tentative extrapolation of them to the threshold as presented in I, have been a matter for some discussion, mostly in connection with the very controversial value of the Panofsky ratio ([6]). Thus the main purpose of the present publication is to show in detail the results themselves and how they have been obtained and analysed. It was felt that this analysis (based on the impulse approximation) was appropriate. It is also our opinion that at the photon energies considered by us the uncertainties associated with the binding of the neutron were always within the limits of the experimental errors and therefore negligible. The extrapolation toward the threshold of the free neutron photo-pion production is then a matter for speculation (*) but does not involve the reliability of the experimental results.

2. – Experimental procedure and data.

The experimental apparatus was identical to that used by BERNARDINI and GOLDWASSER ([7]) in their « pellicles » experiment and already described in

([5]) A. BENEVENTANO, G. BERNARDINI, G. STOPPINI and L. TAU: *CERN Symposium*, **2**, 259 (1956). For other references see ref. ([4]).

([6]) See reports on the Rochester Conference 1955-56, together with other papers on the relationship between Panofsky's ratio and S-wave scattering of pions in hydrogen.

(*) See, for example, M. J. MORAVSIČ: *Nuovo Cimento*, **7**, 442 (1958).

([7]) G. BERNARDINI and E. L. GOLDWASSER: *Phys. Rev.*, **94**, 729 (1954).

some detail in I. The only difference was the target which was filled by liquid deuterium instead of hydrogen. One of the characteristics of the Illinois hydrogen target ([8]) is the easiness with which deuterium is condensed in its appendix (the real beam-target). The target was irradiated by the bremsstrahlung beam of the Illinois 300 MeV betatron. The betatron (and maximum photon) energy was $E_{max} = 300$ MeV. With respect to the photon-beam direction, the blocks of emulsion (in which three 500 μm G-5 pellicles were embedded - see I) were located around the target at 45°, 75°, 105° and 150° (Lab.).

As in I, the energy of the Lab. pions was determined by the position of their decays into the emulsion and then by the range-energy relations.

The main difference and the main problem of this experiment, compared to that made with hydrogen, has been the distinction between positive and negative pions. It is obvious that the distinction must be based on the type of ending, *i.e.* on the fact that positive pions end in a π-μ-e chain while the negative ones end in a « star ». However, the pellicles used in this experiment had been under-developed, as in I. Thus, of the π^+ decays the μ only was clearly visible. Furthermore, the π^- endings associated with the non-visible (ionizing) prongs were practically indistinguishable from the long-range μ's coming from the π decays in flight. Actually here again the decay electron of the μ was invisible.

The following *rules* have been followed in order to fix, without ambiguities, the distinction between positive and negative pions.

a) In making a classification of the π^- stars according to the number of visible prongs, we consider as *effective prongs* only those visible tracks longer than 6 μ. The stars are classified as σ_0, σ_1, etc., according to the number (0, 1, etc.) of visible effective prongs.

b) A safe distinction of the σ_0 was not considered possible on the pellicles exposed during the experiment. Hence the number of σ_0 was determined by an independent experiment, *i.e.* analysing the endings of a 30 MeV π^- beam in Ilford G-5 plates. The exposure was kindly made by Dr. OREAR at the Chicago 450 MeV cyclotron. The details of this exposure and of the correlated σ stars analysis are briefly mentioned in a previous paper ([9]). The corresponding σ-classification (see Table I) was found to be in excellent agreement with the classification done by the Bologna group ([10]). The average of the two classifications, given in the first row of Table I, has been used for the evaluation of the number of σ_0.

([8]) E. A. WHALIN, Jr. and R. A. REITZ: *Rev. Sec. Instr.*, **26**, 59 (1955).

([9]) M. BENEVENTANO, G. BERNARDINI, D. CARLSON-LEE, E. L. GOLDWASSER and G. STOPPINI: *Nuovo Cimento*, **12**, 156 (1954).

([10]) L. FERRETTI and M. MANARESI: *Nuovo Cimento*, **1**, 512 (1955).

TABLE I.

A_0	A_1	A_2	A_3	A_4	A_5
30.2±2.1	32.4±1.3	18.4±1.0	12.8± .8	5.5± .6	1.0± .2

Thus it has been found that

$$A_0 = \frac{N(\sigma_0)}{N_T^-} = .30 \pm .01 . \tag{4}$$

Hence the number N_T^- of the total π^- ending on the pellicles was considered to be

$$N_T^- = N(\sigma \geqslant 1)/(1 - A_0) , \tag{5}$$

where $N(\sigma \geqslant 1)$ means the number of all π^- stars with one or more visible prongs observed in the pellicles. The σ distribution of the pellicles agrees very well with the classification of Table I.

c) In the distinction between positive and negative pions, the main source for a systematic error has been thought to be a possible confusion between a π-μ which does not end in the emulsion, and a σ_1, which prong does likewise.

Whenever it has been possible (for at least 70% of all endings π^+), the tracks have been traced in the adjacent pellicle and their nature was defined by the range and other track characteristics. Of the remaining 30%, at least one half of the tracks showed a prong with a path in emulsion long enough to be identified. The proportion of π-μ present in the remaining doubtful cases was determined by making use of the proportion of them found in the median pellicle. The corresponding error cannot be larger than 2% of the total number of negative pions, if no other error is introduced in the identification of the singly-pronged π^+.

d) To make all preceding criteria really efficient in the discrimination of π^- from π^+ and in the identification of the tracks, all tracks ending in the layers of emulsion near the two surfaces of each pellicle were excluded. The thickness of each layer was 20 μm.

e) In the classification of the singly-pronged π ending, a source of error might be found in some violent scattering near the end of the track. Such scattering in a σ_0 or in a μ may simulate a σ_1 or a π-μ decay.

This error, taking into consideration only those tracks contained within the limits indicated in *d*), was found to be absolutely negligible on the basis of calculation of Coulomb scattering and of analysis done on the Chicago plates used for the σ classification. Furthermore, the σ_1 were frequently indicated by nuclear recoils or by Auger electrons.

Among the now listed criteria the only one not completely objective was the distinction between π-μ decays and σ_1.

This discrimination has been submitted to several checks and all of them turned out to be quite satisfactory. For instance, the ratio $N(\sigma_1)/(\sigma \geqslant 1)$ obtained from the pellicles was compared with the equivalent ratios found in the Chicago plates. The agreement was found to be excellent even in some partial comparisons which were made using either set of pellicles exposed at one angle or definite intervals of energy.

Other checks have been made, for example, to compare the number of σ_1 found in the two external pellicles *and* associated with a nuclear recoil or an Auger electron (*i.e.* the majority of σ_1), with the number of similar tracks found in the median pellicle. As already mentioned, this median pellicle (which allows the tracing of all doubtful escaping tracks) was practically exempt from all uncertainties. Finally, according to (5) the total number of tracks ending in the pellicles without a ($\geqslant 6$ μm) prong, corrected for the evaluated number of $\mu^{\pm}$ originated by π-μ decays in flight, should be and was equal to

$$N_T^- A_0 = N_T^- - N(\sigma \geqslant 1) = N(\sigma \geqslant 1) \frac{A_0}{1 - A_0}, \tag{6}$$

where A_0 is due to the σ classification and $N(\sigma \geqslant 1)$ is the number of σ stars identified in the pellicles.

In conclusion it was believed that the discrimination between positive versus negative pions was so accurate that the corresponding total error in the π^-/π^+ ratio cannot be larger than 3%. This error must be combined with those attached to the corrections introduced to minimize the influence of systematic errors. It is worth-while noticing that most of these errors affect the absolute values of the cross-section, etc., but not the value (function of energy and angle) of the π^-/π^+ ratio. These systematic errors and their corrections were the following.

a) The background due to pions produced in the target walls. This background has been measured and the corresponding fraction of pions subtracted. The measurement of this background was the same as in the hydrogen experiment (see I). The background was found to be about 3% of the total number of pions with a relative error of $\sim 30\%$.

b) The decays in flight. The corresponding correction is ranged between 6% and 3% (see I).

c) Nuclear interaction of the pions before ending. The corresponding correction depends on the range, *i.e.* pion energy, and varies from 0 to 8% (see I).

d) The scanning efficiency. The method of double scanning was systematically applied and the corresponding appropriate correction evaluated, as mentioned in I.

In conclusion, apart from the pure statistiscal error, the over-all error in the π^-/π^+ ratio due to the corrections and discrimination of the sign of pions was estimated to be $\sim 5\%$. A somewhat larger systematic error is expected in the absolute values of the single π^- and π^+ yields.

As already mentioned in the introduction, all experiments concerning interactions with complex nuclei and involving more than two bodies have to face the difficulties originated by the unknown wave-function of the particles inside the nuclear matter. This happens even when attempting to present the experimental results in a simple and direct manner. The problem is still more complicated around the pion threshold, because nuclear momenta inside nuclear matter have the same magnitude as those of the incident photons. Therefore in this paragraph we first present the experimental data in the most empirical way, *i.e.* with reference to the Laboratory frame and making use of the conventional definition of « equivalent photon quantum ».

In the following paragraphs a discussion of these data based on the « impulse approximation » [11] will allow us to derive some more definite and less crude conclusions concerning the, as yet, not well-known reaction (1).

As is known, the cross-section σ_{eq} per equivalent quanta is defined as follows. Let $f(\nu)$ be the incident bremsstrahlung spectrum and ν_{max} the maximum photon energy. The number of equivalent photons is

$$\mathcal{N} = \frac{1}{\nu_{max}} \int_0^{\nu_{max}} \nu f(\nu)\, d\nu \,. \tag{7}$$

Then the number of pions emitted per unit solid angle and unit interval of (pion) energy is

$$\frac{\Delta n_\pi}{\Delta E_\pi \Delta \Omega} = \sigma_{eq} \times N \times \mathcal{N} \,, \tag{8}$$

(11) G. F. Chew and G. C. Wick: *Phys. Rev.*, **85**, 636 (1952); G. F. Chew and M. L. Goldberger: *Phys. Rev.*, **87**, 778 (1952).

where N is the number of deuterons for cm² (in our case $N = 1.57 \cdot 10^{23}$) and Δn_{π^-} is the number of pions detected at the solid angle $\Delta\Omega$ in the limits of the energy interval $E_\pi \to E_\pi + \Delta E_\pi$.

Sometimes one writes $\sigma_{eq} = d^2\sigma_{eq}/dE_{\pi^-}d\Omega$ in order to remember that σ_{eq} has the dimensions cm^{+2} energy^{-1}. Table II gives the values of σ^+_{eq} and σ^-_{eq} at four angles for positive and negative pions from deuterium, as deduced from the analysis of our data (*).

All figures have been corrected for the systematic source of errors which we have mentioned. The quoted error limits include the estimate of the correction uncertainties.

A straightforward comparison of these data with those obtained in hydrogen (4) can easily be made. Actually the two experiments were performed under the same identical conditions, as we have already pointed out.

TABLE II. – (*Units* 10^{-32} cm²/MeV ster).

δ_{lab} / E_γ MeV	45°			75°			105°			150°		
	$\sigma^-_{eq}(D)$	$\sigma^+_{eq}(D)$	$\sigma_{eq}(H)$	$\sigma^-_{eq}(D)$	$\sigma^+_{eq}(D)$	$\sigma_{eq}(H)$	$\sigma^-_{eq}(D)$	$\sigma^+_{eq}(\Omega)$	$\sigma_{eq}(H)$	$\sigma^-_{eq}(D)$	$\sigma^+_{eq}(\Omega)$	$\sigma_{eq}(H)$
170	5.2±.4	3.1±.3	4.6±.4	5.4±.5	3.6±.3	4.0±.3	—	—	—	—	—	—
180	5.7±.4	4.2±.3	5.0±.2	5.4±.3	3.8±.2	4.9±.2	5.2±.4	3.5±.3	4.5±.2	—	—	3.4±.5
190	5.5±.4	4.6±.3	5.3±.4	6.5±.3	4.6±.2	6.1±.3	7.2±.5	5.6±.4	5.2±.4	5.1±.4	3.7±.2	4.1±.2
200	5.6±.4	4.5±.3	5.9±.3	6.8±.3	5.3±.3	6.2±.4	6.8±.5	5.6±.3	6.0±.3	6.1±.4	4.2±.3	4.8±.4
210	5.1±.4	4.4±.3	6.5±.5	6.1±.4	5.8±.3	7.4±.4	7.9±.5	5.5±.4	5.9±.4	6.6±.4	4.8±.3	5.3±.4
220	5.0±.4	4.5±.3	6.0±.6	7.4±.4	5.9±.3	7.8±.4	8.8±.6	6.4±.4	6.5±.4	7.5±.4	5.7±.3	5.8±.4
230	4.5±.8	4.7±.6	—	6.2±.4	6.1±.3	—	7.9±.6	5.9±.4	8.0±.5	9.3±.5	5.6±.3	6.1±.5

Therefore it was very easy to obtain directly from the experimental data of hydrogen the corresponding cross-section σ_H per equivalent quanta. They are also given in Table II. Without anticipation of any particular comment it is clear that σ_H is well comparable with σ^-_{eq} and σ^+_{eq}, and has approximately the same energy trend.

(*) The photon energies listed in Tables II and III are calculated from E_{π^-} using the two-body kinematics. The use of the two-body kinematics will be justified later.

A comparison between σ_{eq}^{+} and σ_H is given in Table III. The ratio between σ_{eq}^{+} and σ_H averaged over all energies and angles is

$$\sigma_{eq}/\sigma_H = 0.812 \pm 0.015 .$$

TABLE III.

E_γ \ θ_{lab}	45°	75°	105°	150°
170	.67±.09	.90±.10	—	—
180	.84±.07	.78±.05	.78±.08	
190	.87±.08	.75±.05	1.08±.11	.90±.07
200	.76±.07	.85±.07	.93±.07	.87±.10
210	.68±.07	.78±.06	.93±.08	.91±.09
220	.75±.09	.76±.05	.99±.09	.98±.09
230	—	—	.74±.07	.92±.09
Average = .812 ± .015				

3. – Discussion on the validity of the impulse approximation and of the (average) two-body kinematics.

3·1. – As already mentioned, the analysis of the experimental data relative to reaction (3) does not allow a simple derivation of the absolute cross-sections corresponding to the free-nucleon reactions (1) and (2). The difficulty is primarily in the kinematics. At each laboratory angle an observed pion of energy E can have been originated by a photon whose energy ν has to be ranged in a rather broad and continuous interval. The shape of this « effective » spectrum of the photons able to produce a pion of definite energy, depends mainly upon the known bremsstrahlung spectrum and the unknown spectrum of nucleon momenta involved in reactions (3).

This spectrum of nuclear momenta is unknown. It lies in the classic problem of the nuclear forces. A clear indication of this difficulty is given, for instance, by the existence of the deuteron photodisintegration,

(9) $$\gamma + D \to p + n .$$

This reaction, with a smaller but comparable cross-section, competes with reaction (3). Furthermore, at photon energies $\nu \gtrsim \mu$ (*) reaction (9) is essentially a mesonic process and it is theoretically much less simple and less understood than reactions (1) and (2). In other words, a theoretical analysis of reaction (9) should automatically involve the analysis of reaction (3) and *vice versa*, because it is the same photon-deuteron interaction which originates the processes (3) and (9).

However, one may be inclined to assume that reaction (3) represents a typical process to which one may apply the « *impulse approximation* ». As it is well known, this is a method of approximation for the calculations of transition rates induced by a fast incident particle on a many-body target as, for instance, a nucleus. This method has been clearly and extensively discussed by CHEW and WICK, and by CHEW and GOLDBERGER ([11]).

The fundamental assumption is that the transition amplitudes of the various single nucleons in the target (in our case two) are linearly superposable. The conditions for its validity are essentially the following: i) negligible interaction of the emitted particles with the other « spectator » nucleons; ii) a collision time which is short in comparison with the average transit time of a nucleon inside the nucleus. Both conditions seem to be very well verified for the photoproduction of pions by deuterium. Some experimental facts also support this point of view. They are the following: i) The Stanford experiments ([12]) show that at photon energies $\nu \gtrsim \mu$ the *average* distance of the two nucleons is large with respect to the photon $\lambda\!\!\!^{-}$ and the Compton pion wave length $1/\mu$. ii) The *average* interaction energy between the two nucleons is small in respect of ν. iii) The final state of reaction (3) is identical to that of scattering of pions in deuterium at moderate energies. The recent work of LEDERMAN *et al.* ([13]) shows the validity of the impulse approximation for this kind of scattering. However, a more stringent analysis of the conditions required for the validity of the impulse approximation may raise some doubt about the plain application of this method to the analysis of the results of an experiment concerning reaction (3). With particular reference to the CHEW and GOLDBERGER paper, two points deserve some consideration. The first concerns the so-called « *multiple interaction* » of the emitted photo-pion. This depends upon the amplitude and phase-shifts of the outgoing pion wave with respect to the « spectator » nucleon. That is, this multiple interaction depends on the pion energy, the relative two-nucleon momentum $|\boldsymbol{k}| = |(\boldsymbol{K}_1 - \boldsymbol{K}_2)/2|$ and the relative nucleon distance $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. Near photo-pion threshold ($\nu \simeq \mu$) the error due to the multiple scattering is certainly negligible if small values of $|\boldsymbol{r}|$ and

(*) Here and in the following, $\hbar = c = 1$; μ is the pion rest energy.

([12]) R. HOFSTADTER and R. W. MCALLISTER: *Phys. Rev.*, **98**, 217 (1955); R. HOFSTADTER and R. W. MCALLISTER: *Phys. Rev.*, **103**, 1454 (1956).

([13]) L. M. LEDERMAN and K. C. ROGERS: *Phys. Rev.*, **105**, 247 (1957).

correspondingly very high values of $|k|$ are excluded ([14]). The second point is that in a more rigorous approach the energy which must be compared with that of the interacting photon is the actual potential $U(r)$ and not the binding energy ε. The corresponding error depends upon the correlated probability $\Psi^*(r)\Psi(r)$ of finding the two nucleons at the actual distance r.

The form of $U(r)$ and $\Psi(r)$ is a classic problem and all reasonable approaches, including the most recent (that of GARTENHAUS ([15]), based on the Chew and Low theory), bring to a potential wall $U(r)$ which is rather extended, but deep ($U(r) \geqslant 20$ MeV) *only* in a fairly narrow region at an average distance of less than one fermi. Furthermore, as is indicated by several experimental facts, the more central part ($r \simeq 0.5$ fermis) is then practically excluded by a sharply repulsive core. This kind of potential favours the « tunnel effect », *i.e.* the broadening of the wave-functions toward large values of r. Consequent upon this fact, at distances $r \gtrsim 1/\mu$ the corresponding wave-functions are quite independent from the particular form chosen to represent $U(r)$. The experimental results of MCINTYRE and HOFSTADTER ([16]) seem to be fairly consistent with the behaviour common to all suggested types of deuteron wave-functions for $r \gtrsim 1/\mu$. These results also exclude any very bad behaviour of the deuteron wave-function for distances $r \lesssim 1/\mu$. Again, the interpretation of the Stanford experiments requires the introduction of the nuclear repulsive core. Therefore, the interaction pictured by such a potential wall $U(r)$ makes doubtful the application of the impulse approximation for $r \leqslant 1/\mu$, but fortunately the probability of finding the two nucleons at such a distance is quite small. The conclusion seems to be that in doing the analysis of reaction (3) the impulse approximation may be considered valid, at least up to nucleon momenta corresponding to the distance $r \gtrsim 1/\mu$. Furthermore, for nucleon distances $r \gtrsim 1/\mu$, any type of wave-function adjusted to the deuteron binding energy and to the triplet scattering length is supposed to represent a good approximation of the reality. In practice, all wave-functions are equivalent for $r \gtrsim 1/\mu$, and for the sake of simplicity one may well use the Hulthén wave-functions.

The aim of the following is to make a plausible use of the previous conclusions in the evaluation of the spectrum of the photons correlated with a pion $\pi(E_\pi, \vartheta)$ of energy E_π observed at a laboratory angle ϑ. It will be shown that toward the high photon energies there is a « tail » of this spectrum; but this tail is rather small and it remains so even in view of some drastic energy dependence of the matrix elements. Hence, it may be seen that the inclusion or

([14]) G. CHEW and H. LEWIS: *Phys. Rev.*, **84**, 779 (1951).

([15]) *H.* GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955).

([16]) *Proceedings of the VII Annual Rochester Conference on High-energy Physics*, (1957).

suppression of this tail does not appreciably change the energy correspondence existing between the photon spectrum now mentioned and $\pi(E_\pi, \vartheta)$. Next we will show that the inclusion of the interval $0 \leqslant r \leqslant 1/\mu$ and the use of the conventional Hulthén wave-functions in this interval do not appreciably affect the results obtained by the impulse approximation.

3·2. – The photoproduction of a single pion from deuterium can be described as a transition from an asymptotic initial state $|i\rangle$ = deuterium plus a photon, to an asymptotic final state $|k\rangle$ where: i) the photon has been destroyed, ii) a (free) pion has been created, and iii) the initial proton has been converted into a neutron or *vice versa* and the two final nucleons are in a state of relative momentum k. It is conventional to make use of the laboratory system. The parameters involved in the reaction are:

1) the momentum $\boldsymbol{\nu}$ and the energy ν of the incident photon, in units μ;
2) the pion momentum $\boldsymbol{\eta}$ and its total energy ω, in units μ;
3) the final momentum $\boldsymbol{k} = (\boldsymbol{K}_1 - \boldsymbol{K}_2)/2$ of the two nucleons, in units μ;
4) their total momentum $\boldsymbol{D} = \boldsymbol{K}_1 + \boldsymbol{K}_2$, in units μ;
5) the co-ordinate $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ of the C.M. and the relative distance $(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \boldsymbol{r}$ of the two nucleons. The neutron-proton mass difference Δ is neglected because the pion momentum here considered is always $\eta^2 \ll \Delta/\mu$. However, even at energies ($E_\gamma \geqslant 170$ MeV) not too close to threshold, the binding energy $D_0 = -\varepsilon$ of the deuteron should be taken into account.
6) The nucleon total energy is simply $E_i = \frac{1}{2}(K_i^2/M) + M$.

The energy momentum conservation yields

$$\boldsymbol{\nu} - \boldsymbol{\eta} = \boldsymbol{D}\,, \tag{10}$$

$$-\varepsilon + \nu = \frac{1}{2M}(K_1^2 + K_2^2) + \omega\,, \tag{10'}$$

i.e.

$$\varepsilon + \omega + \frac{k^2}{M} + \frac{(\boldsymbol{\nu} - \boldsymbol{\eta})^2}{4M} - \nu = 0\,. \tag{11}$$

Obviously (10), which does not imply $\boldsymbol{k}$, is not enough to define uniquely the kinematics of the reaction (*).

(*) The two-body relationship between the meson momentum and its angle with respect to the incident photon

$$2\nu(\omega - \eta\cos\vartheta) = 1 + 2M(\nu - \omega)\,, \tag{A}$$

In other words, the emission of a pion of momentum ($\boldsymbol{\eta}$, $\mathrm{d}\boldsymbol{\eta}$) due to a photon of momentum $\boldsymbol{\nu}$ must be associated with a final nucleon state having the total momentum ($\boldsymbol{D}$, $\mathrm{d}\boldsymbol{D}$) and a final interval nucleon momentum ($\boldsymbol{k}$, $\mathrm{d}\boldsymbol{k}$), correlated by the condition (11). Hence, if E_i and E_k are the *total* energies of the initial and final state respectively, the transition rate from $|i\rangle$ to $|k\rangle$ is given ($\hbar = c = 1$), by

$$\mathrm{d}W_k = 2\pi |H_{i,f}|^2 \delta(E_i - E_k) \frac{1}{(2\pi)^3} \mathrm{d}\boldsymbol{\eta}\, \mathrm{d}\boldsymbol{D} . \tag{12}$$

Now in order to average over all possible $|k\rangle$ states, instead of the relative energy $\varepsilon_k = k^2/M$ one may conveniently use as integration variable the relative momentum k (extended over half of the k_f space to take into account the identity of the two nucleons). Let us call

$$\xi_0 \chi_{1,m} U_i = \Psi_i \tag{13}$$

the initial deuteron wave-function and

$$\psi_k = \xi_\pm \begin{pmatrix} \chi_{1,m} & U_a \\ \chi_0 & U_s \end{pmatrix} \exp i[\boldsymbol{D}\cdot\boldsymbol{R}] \tag{14}$$

the final nucleon state split in the symmetric and antisymmetric spatial wave-functions. Here ξ and χ are the usual isospin and spin singlet and triplet

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now becomes

$$\cos\vartheta = \frac{\nu^2 + \left[\nu - \left(\frac{k^2}{M} + \frac{D^2}{4M}\right)\right]^2 - (1+D)^2}{2\nu \left\{\left[\nu - \left(\frac{k^2}{M} + \frac{D^2}{4M}\right)\right]^2 - 1\right\}^{\frac{1}{2}}} \tag{B}$$

and at a definite angle ϑ all values of η are possible provided that

$$\nu - \left(\varepsilon + \omega + \frac{D^2}{4M}\right) = \frac{k^2}{M} \gtrsim 0 , \tag{C}$$

that is, depending upon the final value of k.

If one introduces the threshold photon energy ν_t this can be approximatively expressed by

$$\sqrt{M(\nu - \nu_t)} \gtrsim D . \tag{D}$$

The average value of k^2/M is $D^2/4M$ ((see CHEW and LEWIS, ref. [14])) .

eigenfunctions. Then the probability for the photo-emission of a pion $(\boldsymbol{\eta}, \mathrm{d}\boldsymbol{\eta})$ from deuterium can be written

$$\mathrm{d}\sigma = \frac{1}{(2\pi)^2}\,\mathrm{d}\boldsymbol{\eta}\int \mathrm{d}\boldsymbol{k}\,|H_{ki}|^2\,\delta(E_i - E_k) = \tag{15}$$

$$= \frac{1}{(2\pi)^2}\,\mathrm{d}\boldsymbol{\eta}\int \mathrm{d}\boldsymbol{k}\,\Big|\int \mathrm{d}\boldsymbol{R}\langle\psi_k|\,T\,|\psi_i\rangle\Big|^2\,\delta(E_i - E_k)\,.$$

The overlapping integral with respect to $\boldsymbol{R}$, eliminates, according to (10) one of the two degrees of freedom $\boldsymbol{\eta}$ and $\boldsymbol{D}$. For a fixed photon momentum ν the double integral over $\boldsymbol{R}$ and $\boldsymbol{k}$ gives (apart from a normalization factor) either the probability of finding a pion of momentum $(\boldsymbol{\eta}, \mathrm{d}\boldsymbol{\eta})$ or the corresponding nucleon recoil $(\boldsymbol{D}, \mathrm{d}\boldsymbol{D})$. For comparison with an experiment where only pions have been detected, it is more convenient to use (15) instead of the equivalent but somewhat simpler differential expression function of $\boldsymbol{D}$. To make explicit the Laboratory solid angle $\mathrm{d}\Omega$ (where the pion has to be observed) and the pion energy ω one writes

$$\begin{cases} \mathrm{d}\boldsymbol{\eta} = \eta^2\,\mathrm{d}\eta\,\mathrm{d}\Omega_\pi\,, \\ \mathrm{d}\eta = \dfrac{\omega}{\eta}\,\mathrm{d}\omega\,, \end{cases} \tag{16}$$

and finally has

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega_\pi\,\mathrm{d}\omega} = \frac{1}{(2\pi)^2}\,\eta\omega\int \mathrm{d}\boldsymbol{k}\,|H_{ki}|^2\,\delta(E_i - E_k)\,, \tag{17}$$

which can be considered as the expression of the energy spectrum $\varphi(\omega)$ of the pions emitted at $\mathrm{d}\Omega_\pi$ by the monochromatic photon ν. The internal motion of the nucleons into the deuterium is the source of such an energy spread.

The matrix element $\langle\Psi_k|\,T\,|\Psi_i\rangle$ has been evaluated with the use of the impulse approximation (or equivalent) by several authors [17].

3·3. – The following is simply a brief reminder of these calculations. They give what can be called the « effective » photon spectrum able to produce a pion of energy E_π at the angle ϑ. All desirable details about this type of calculation can be found in the previously quoted papers. Our purpose is to show the use made of them in the estimate of the kinematic indefinition due

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[17] G. MORPURGO: *Nuovo Cimento*, **7**, 855 (1950); M. LAX and H. FESHBACH: *Phys. Rev.*, **81**, 189 (1951); **88**, 509 (1952); G. CHEW and H. LEWIS: *Phys. Rev.*, **84**, 779 (1951); S. MACHIDA and T. TAMURA: *Prog. Theor. Phys.*, **6**, 572 (1951); W. THIRRING: *Helv. Phys. Acta*, **26**, 465 (1953); A. M. BALDIN: *Comp. Rend. CERN Symposium*, **2**, 272 (1956); S. PENNER: *Phys. Rev.*, **105**, 1113 (1957).

to the motion of the nucleons inside the deuterium. According to the fundamental assumption of the impulse approximation the scattering matrix T is approximated by

$$T = T_1 + T_2 = \sum_1^2{}_j \tau_j^{\pm}(\boldsymbol{\sigma}_j \cdot \boldsymbol{K}^{\pm} + L^{\pm}) \exp i[\boldsymbol{r}_j \cdot (\boldsymbol{\nu} - \boldsymbol{\eta})] , \tag{18}$$

where each T_j refers to the nucleon j as it would on a *free* nucleon. The operators written in (18) are:

i) the usual isospin and spin operators

$$\tau_j^{\pm} = (\tau_x \pm i\tau_y) \text{ and } \boldsymbol{\sigma} ;$$

ii) an operator $\boldsymbol{K}$ able to change the spin state of the nucleon system; and

iii) an operator L independent from σ which induces photomeson production.

Both $\boldsymbol{K}$ and L depend on $\boldsymbol{\nu}$, $\boldsymbol{\eta}$, $\boldsymbol{k}$, and on the photon polarization. They are different when τ^- is replaced by τ^+, *i.e.* they depend upon the sign of the emitted pion. The π^-/π^+ ratio is due to this difference.

With the above approximation of the final states, the number of pions produced in the pion energy interval $\Delta\omega = \omega_2 - \omega_1$ in the solid angle $\Delta\Omega$ and at the angle ϑ with the direction of the photon, is given by (see LAX and FESHBACH, l. c.),

$$n^{\pm}(\omega_1, \omega_2, \vartheta)\Delta\Omega = \int_{\nu_t}^{\nu_{\max}} \mathrm{d}\nu \int_{\omega_t}^{\omega_2} \eta\omega f(\nu) \{ H_1^{\pm} I_1 - H_2^{\pm} I_2 \} \mathrm{d}\omega , \tag{19}$$

where: $f(\nu)$ is the differential spectrum of the incident photon beam;

$$H_1^{\pm} = |K^{\pm}|^2 + |L^{\pm}|^2; \quad H_2^{\pm} = \tfrac{1}{2}|K^{\pm}|^2 + |L^{\pm}|^2;$$

$$I_1 = \int |C(\boldsymbol{K} - \boldsymbol{K}_0)|^2 \, \delta(E_i - E_k) \, \mathrm{d}\boldsymbol{K};$$

$$I_2 = \int C(\boldsymbol{K} - \boldsymbol{K}_0) \, C(\boldsymbol{K} + \boldsymbol{K}_0) \, \delta(E_i - E_k) \, \mathrm{d}\boldsymbol{K} .$$

In the integrals I_1 and I_2, $\boldsymbol{K}_0$ is the momentum of the final nucleon if initially the nucleon was free, and $C(\boldsymbol{K})$ is the wave-function of the deuterium in the momenta representation, *i.e.* $C(\boldsymbol{K})$ is Hultén's function.

It is convenient to write for (19)

$$n^{\pm}(\omega_1, \omega_2, \vartheta)\Delta\Omega \propto \Delta\Omega \int_{\nu_t}^{\nu_{\max}} F_{\Delta\omega}(\nu)\,\mathrm{d}\omega\,, \tag{20}$$

where

$$F_{\Delta\omega}(\nu) = f(\nu) \int_{\omega_1}^{\omega_2} \eta\omega\,\{H_1^{\pm} I_1 - H_2^{\pm} I_2\}\,\mathrm{d}\omega\,, \tag{21}$$

may be considered as the « efficiency » function for the photons in the interval $(\nu, \mathrm{d}\nu)$. In other words $F_{\Delta\omega}(\nu)$ expresses the probability that those photons create a pion into the energy interval $\Delta\omega = \omega_2 - \omega_1$ and the solid angle $\Delta\Omega$.

The evaluation of (21) would require the knowledge of $H_1^{\mp}$ and $H_2^{\pm}$. To avoid all uncertainties and, as will be seen, almost useless complications involved in the explicit definition of the operators $\boldsymbol{K}$ and L, a simple numerical evaluation of $F_{\Delta\omega}(\nu)$ has been performed considering two extreme cases. In one case $H_1^{\pm}$ and $H_2^{\pm}$ have been considered constant in respect to ν: in the other one as varying proportionally to $(\nu - \nu_t)$. These two drastic assumptions are equivalent to saying that in the first case the S-wave is largely the

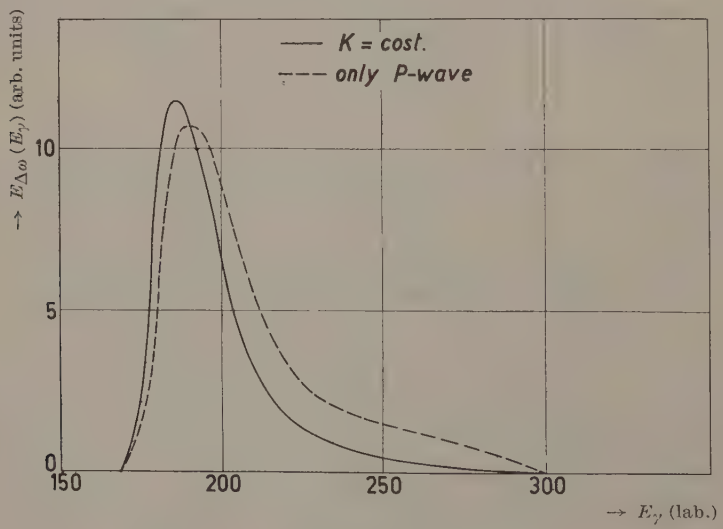

Fig. 1.

dominant, while in the second the P-wave is chiefly the leading amplitude. The two typical curves $F_{\Delta\omega}(\nu)$ evaluated to show this influence of the possible energy dependence of the matrix elements are given in Fig. 1. They refer to

$\vartheta = 75°$ (Lab) and $\Delta\omega = (180 \div 190)$ MeV. These curves (*) deserve a few comments. i) For both curves the position of the maximum and the average photon energy are quite close to those of the two-body kinematics with a target-nucleon at rest in the Lab. frame. This fact has already been emphasized by CHEW and LEWIS and by LAX and FESHBACH. ii) As expected they are asymmetric but the « tail » toward the high energies is not too relevant. iii) In any case these tails are due to the inclusion of initial high nucleon momenta. As pointed out by WILSON [18], it is likely that most of these high momenta are involved in reaction (9) but excluded from the process of the photoproduction of real pions by deuterium.

In concluding, *when doing the analysis of the data concerning reaction* (3), *one may consider as justified the use of the two-body kinematics as a fair approximation for the evaluation of the* (*average*) *photon energy which corresponds to a pion emitted at an angle* ϑ, *with a definite energy in the Lab. system.* The errors involved in this approximation will be quantitatively estimated in the following section. It will be seen that in the energy range considered, they are always smaller than the experimental errors.

3·4. – Because it was clear that the previous discussion did not follow a quite rigorous line of reasoning, and because the experimental facts supporting the application of the impulse approximation were somewhat indirectly correlated to the problem under discussion, it was considered interesting to find a more direct experimental fact supporting the arguments developed in the preceding paragraph. This fact refers to the high energy « tail ».

As already mentioned, the deuterium experiment was made parallel to that of hydrogen (see I). At $\vartheta = 150°$ (Lab.), the maximum energy of the pions produced in hydrogen by a photon ($E_{\gamma\max} = 300$ MeV), is 62.5 MeV. The endings of these pions were observable in our pellicles. The same was applicable for the endings of the positive and negative pions emitted at the same angle by the deuterium target. A comparison between the high energy edges of the two-pion spectra was thus possible.

The results are summarized in Table IV. The range intervals and the corresponding initial kinetic energies of the pions are listed in the first and second columns respectively. The third, fourth and fifth columns give the number of observed pions (into the median pellicle, see Sect. 2 of this paper) corrected only for the efficiency of the scanning. The remaining columns give the numbers $n = \Delta n/(\Delta E_{\pi} \cdot N_T)$, *i.e.* the relative number of pions per unit energy interval.

(*) Other $F_{\Delta\omega}(\nu)$, all corresponding to the case of a predominant S-wave, are given in Figs. 4 and 5.

(18) R. WILSON: *Phys. Rev.*, **104**, 218 (1956).

TABLE IV.

ΔR (mm)	ΔE_π (lab. MeV)	Δn_H^+	Δn_D^+	Δn_D^-	n_H^+	n_D^+	n_D^-
4.45÷48.5	5.89÷61.9	37	45.5	81	.186	.116	.132
48.5÷52.5	61.9÷64.7	19	21	41.5	.103	.057	.073
52.5÷56.5	64.7÷67.4	2	15	31.5	.011	.043	.057
56.5÷60.5	67.4÷70.1	4	23.5	22	.023	.066	.040
60.5÷64.5	70.1÷72.7	3	15	16	.018	.044	.030
64.5÷68.5	72.7÷75.7	1	10	8	.006	.030	.016
68.5÷72.5	75.2÷77.7	0	1.5	4	.000	.005	.008

A rough comparison with our calculations is then quite easy because at these high photon energies $I_2 \ll I_1$ and $H_2 I_2$ can be dropped in (21). Furthermore one may assume

$$f(\nu) = \begin{cases} \dfrac{1}{\nu} & \nu \leqslant 300 \text{ MeV} \\ 0 & \nu > 300 \text{ MeV}. \end{cases} \tag{22}$$

In the extreme case

$$|K|^2 + |L|^2 \simeq \text{const}$$

one would have

$$n(\omega, \omega + \Delta\omega, \vartheta = 150°) \propto \eta \int_{\nu_t}^{\nu=300 \text{ MeV}} I_1 \frac{d\nu}{\nu}.$$

In Fig. 2, the function $(\omega, \omega+\Delta\omega, \vartheta = 150°)$ is plotted (solid line) together with the values of n_H^+, n_D^+ and n_D^- as derived from Table IV. In spite of the

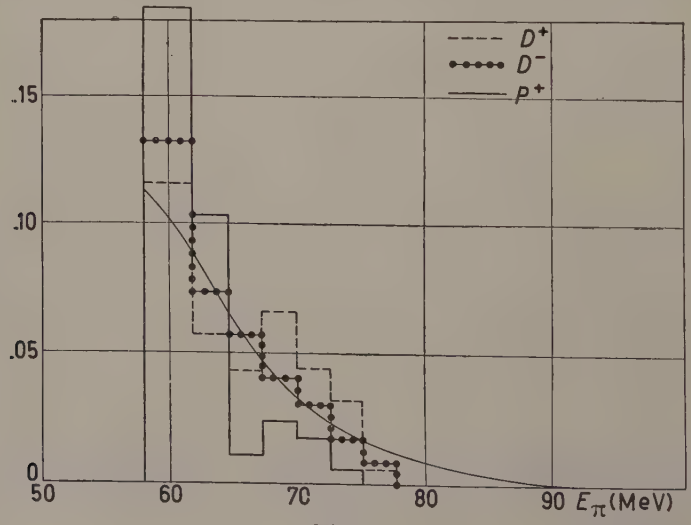

Fig. 2.

rather meager statistics it is evident that, if anything, the experimental « tail » is less than that evaluated with $|K|^2+|L|^2$ const.

Around $\nu \simeq 300$ MeV, a substantial contribution of the P-amplitude is expected. As is shown in Fig. 2, the tail should then be larger than that evaluated in the extreme opposite case $|K|^2+|L|^2 \simeq$ const. Thus in spite of the poor statistics it seems that the experimental tails are indeed not larger than those introduced in the evaluation of the efficiency functions $F_{\Delta\omega}(\nu)$. Independently from any speculation connected with the Wilson argument, this fact tells us that the actual errors due to the application of the two-body kinematics are probably smaller than those derived from the use of these functions.

4. – The differential cross-sections of reaction (3).

Once it is proved that the simple two-body kinematics with nucleons at rest in the Laboratory represents a good approximation, one can then make better use of the experimental data, deriving from them the cross-sections of deuterium *averaged* over definite but rather narrow photon energy intervals. These *deuterium* cross-sections, as a function of the photon energy and pion-photon angle (Lab.), are given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{D}}^{\pm} = \frac{n_\pi^{\pm}}{N_{\mathrm{D}}\Delta\Omega(\mathrm{d}\mathcal{N}/\mathrm{d}\nu)(\mathrm{d}\nu/\mathrm{d}\omega)\Delta\omega}\,, \tag{23}$$

where $n_\pi^{\pm}$ is the number of $(\pm)$ pions observed into the energy interval $\Delta\omega$ (*), $\Delta\Omega$ is the solid angle subtended by the pellicles, and N_{D} the number of deuterium atoms per cm² crossed by the photon spectrum. The photon energy interval is $\Delta\nu = (\mathrm{d}\nu/\mathrm{d}\omega)\Delta\omega$. Having once fixed (arbitrarily) $\Delta\nu$ taking into account the neutron-proton mass difference (+), one determines the corresponding $\Delta\omega$ by the two-body kinematics. The cross-section calculated by (23) is given in Table V. The table is divided according to the Lab. angles and the related photon energy intervals. In the first column are the photon averaged energies; the corresponding pion energies are then specified for each angle.

The quoted errors take into account all the estimated experimental sources of uncertainty, except that due to the division of the emulsion into sections corresponding to the chosen photon intervals. Due to the kinematics this

(*) Corrected for systematic source of errors as indicated in Sect. 2.

(+) At photon energies higher than 170 MeV the shift in the pion energy due to this mass difference is less than 5%.

TABLE V. – (*Units* 10^{-30} cm²/ster).

ϑ_{lab} / E_γ	45°		75°		105°		150°	
MeV Lab	$d\sigma^+/d\Omega^*$	$d\sigma^-/d\Omega^*$	$d\sigma^+/d\Omega^*$	$d\sigma^-/d\Omega^*$	$d\sigma^+/d\Omega^*$	$d\sigma^-/d\Omega^*$	$d\sigma^+/d\Omega^*$	$d\sigma^-/d\Omega^*$
170	3.5± .3	5.9± .5	4.8±.4	7.4±.6				
180	5.3± .4	7.3± .5	5.3±.3	7.6±.4	4.8± .6	8.9±1.0		
190	6.4± .4	7.7± .5	6.9±.3	10.2±.5	8.2± .8	10.1±1.0	7.3±.5	10.4± .7
200	7.3± .5	8.7± .6	8.4±.4	10.7±.5	9.2± .9	10.4±1.1	8.3±.6	12.2± .7
210	6.8± .5	7.9± .7	9.6±.5	10.1±.6	8.4± .9	13.5±1.3	9.6±.6	13.2± .8
220	7.3± .6	8.2± .7	10.2±.5	13.1±.7	12.2±1.1	16.4±1.5	11.9±.6	15.3± .9
230	7.3±1.0	7.3±1.3	11.0±.6	11.8±.7	12.6±1.5	14.3±1.8	11.7±.7	19.8±1.1

error is relatively larger at the backward angles. At these angles the sections of the pellicles to which we now refer were rather narrow and their edges not too well defined.

To minimize the corresponding error at the backward angles (105°, 150° Lab.) the photon intervals have a width of 20 MeV. At the more favourable forward angles (45° and 75°), intervals of 10 MeV have been considered.

It may be that in themselves the cross-sections of Table V do not have any very significant interest. Their interest lies mainly in the possibility of a comparison with the cross-sections of the really free nucleons.

5. – « Attenuation » with respect to the free proton.

5·1. – The « *attenuation* » should be understood as the effect of i) the exclusion of the (over-all) symmetric nucleon final states by the Pauli principle; ii) the binding between the two nucleons which (apart from the occurrence of reaction (9)) imposes a limitation of kinematical origin. It is essentially due to the condition (C) of footnote page 1120. This limitation is sensitive only at fairly low energies ($E_\gamma \gtrsim 200$ MeV).

Both these causes of attenuation are independent from the occurrence of reaction (9) and are decreasing asymptotically when (beyond threshold) the photon energy increases. Instead, the cross-section for reaction (9) which far

from threshold is the main « attenuation » cause, is rather flat and almost constant in a wide energy interval beginning at $E_\gamma \simeq 100$ MeV and extending up to $E_\gamma \simeq 400$ MeV ([19]). Even around the threshold for pion production, it does not reveal any competition from the other photon-induced processes. This fact seems consistent with the main assumption of the Wilson argument ([18]).

With reference to formula (17), the *attenuation* can be approximately written as follows:

$$1 - A(\nu, \vartheta) = \frac{\mathrm{d}\sigma_{\mathrm{D}}^{+}}{\mathrm{d}\sigma_{\mathrm{H}}} = \frac{\int \mathrm{d}\eta \int \mathrm{d}\boldsymbol{k} \, |H_{k,i}|^2 \delta(E_i - E_k)}{\int \mathrm{d}\eta [|K_0|^2 + |L_0|^2] \delta_{\mathrm{H}}}, \tag{24}$$

where

$$\delta_{\mathrm{H}} = \delta\{[M^2 - (\boldsymbol{\nu} - \boldsymbol{\eta})^2]^{\frac{1}{2}} + \omega - \nu - M\} \simeq \delta\left\{\nu_0 - \omega + \frac{(\boldsymbol{\nu} - \boldsymbol{\eta})^2}{2M}\right\}.$$

Formula (24) is the more convenient for a comparison with the experimental results because the cross-sections are referring to the pion momentum $\boldsymbol{\eta}$. According to the basic idea of impulse approximation, the operators K_0 and L_0 of the matrix element at the denominator are supposed to be the same as the K and L occurring in the numerator. As far as one makes a clear distinction between reaction (3) and reaction (9) (*), the use of the same operators as both the numerator and denominator of (24) is certainly a good approximation and fits into the frame of the preceding discussions. When the finite size of the chosen experimental photon intervals is taken into account according to (19), (21) and (22), instead of (24) one has to write

$$\left\langle \frac{\mathrm{d}\sigma_{\mathrm{D}}^{+}}{\mathrm{d}\sigma_{\mathrm{H}}} \right\rangle \simeq \frac{\int_{\nu_t}^{\nu_{\max}} \mathrm{d}\nu \int_{\omega}^{\omega + \Delta\omega} \eta\omega f(\nu) \{H_1^{+} I_1 - H_2^{+} I_2\} \mathrm{d}\omega}{\int_{\Delta\nu} \int_{\omega}^{\omega + \Delta\omega} \eta\omega \{|K|^2 + |L|^2\} \delta\left\{\frac{(\boldsymbol{\nu} - \boldsymbol{\eta})^2}{2M} + \omega - \nu\right\} f(\nu) \, \mathrm{d}\nu \, \mathrm{d}\omega}, \tag{25}$$

where $\Delta\nu$ is the photon interval corresponding to $\Delta\omega$ and $\langle\ \rangle$ means the average over the same interval.

Table VI shows the values of $\langle 1 - A(\nu) \rangle$ at 45° and 75°.

Table VII presents the same comparison for 105° and 150°. The difference in the energies of the chosen intervals are again due to the limitation imposed by the kinematics of the laboratory frame. In the same tables are also listed

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([19]) L. Allen: *Phys. Rev.*, **98**, 705 (1955); B. D. Schriever and A. O. Hanson: *Phys. Rev.*, **101**, 377 (1955); J. Keck, R. M. Littauer, G. K O'Neill, A. M. Perry and W. M. Woodward: *Phys. Rev.*, **93**, 827 (1954); J. Keck and A. Tollestrup: *Phys. Rev.*, **101**, 360 (1955).

(*) Actually at the lowest order the operator of (9) contains (besides a small term in e) terms in ef^2 while reaction (3) has matrix elements in ef.

TABLE VI.

ϑ_{lab} / $E_{\gamma\ lab}$	45°			75°		
	Calc.		Obs.	Calc.		Obs.
	$K^2=0$	$L^2=0$		$K^2=0$	$L^2=0$	
180 (165÷195)	0.53	0.79	0.80±.04	0.61	0.81	0.77±.03
210 (195÷225)	0.55	0.80	0.78±.04	0.66	0.83	0.80±.03

the calculated values of $1 - A(\nu, \vartheta)$ in the extreme cases $|K|^2/|L|^2 \simeq 0$ and $|L|^2/|K|^2 \simeq 0$. As expected, the comparison is consistent with the assumption that the interaction is mainly due to the spin-flipping term K. This is simply

TABLE VII.

ϑ_{lab} / $E_{\gamma\ lab}$	105°			150°		
	Calc.		Obs.	Calc.		Obs.
	$K^2=0$	$L^2=0$		$K^2=0$	$L^2=0$	
195 (185÷205)	0.73	0.84	0.91±0.08	0.74	0.84	0.91±.07
215 (205÷235)	0.75	0.86	0.87± .06	0.77	0.86	0.95±.06

a direct confirmation of what is today a well-known fact: at low energies the leading term of the photoproduction of charged pions is the S amplitude due to an electric dipole transition. As shown by BALDIN (ref. [17]) the high frequency of the « forks » observed by ADAMOVIČ *et al.* [20] (*) also proves directly that, at least up to $E_\gamma = 200$ MeV, $|L|^2/|K|^2 \leqslant 0.10$. Our results have only the advantage of giving some indication concerning the angular dependence

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[20] M. I. ADAMOVIČ, V. I. VEKSLER, G. V. KUZMIČEVA, V. G. LARIONOVA and S. P. HARLAMOV: *Comp. Rend. CERN Symposium*, **2**, 265 (1956).

(*) See in particular Table III on page 270 of paper Ref. [20].

of $d\sigma/d_{\Omega})_D/d\sigma/d_{\Omega})_H$. This would be of some help in the following discussion, particularly in view of the crudeness of the calculations (*).

At higher energies the non-flipping amplitudes becomes clearly evident mostly for the enhancement of the M_{33} magnetic pole absorption (21). This fact can be seen, for instance, in the paper by HAGERMAN *et al.* (3). If use is made of the two-body kinematics, the results presented by these authors show that at $E_\gamma = 220$ MeV, $|L|^2$ is still very small while at $E_\gamma = 270$ MeV the influence of the non-flipping term $|L|^2$ is already quite significant (+).

Finally, Fig. 3 shows the ratio of the total cross-sections as obtained averaging over the four angles investigated in this experiment and over the

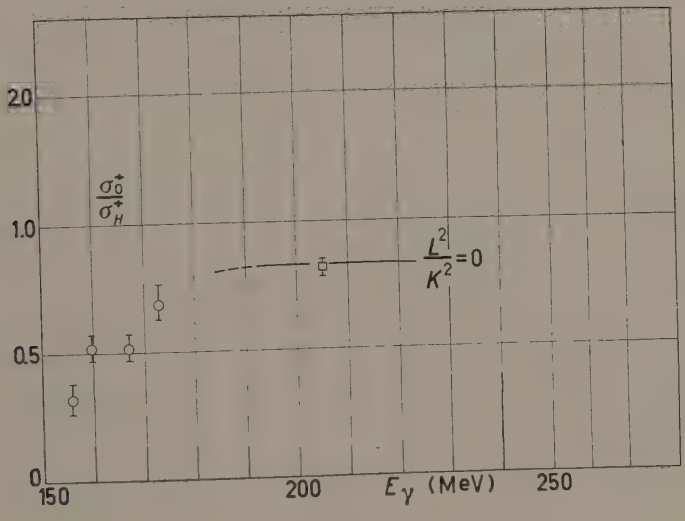

Fig. 3.

energy interval (185 ÷ 225) MeV, plotted besides the lower energy points due to Penner (×). The solid line gives the calculation of σ_D^+/σ_H when $L^2/K^2 \simeq 0$. It is significant that the observed attenuations are in agreement or even smaller than the calculated ones. Of course, the error limits leave an adequate clearance for the Compton effect and other very small cross-sections.

(*) Our efficiency curves are very broad for the emission of pions at backward angles.

(21) M. GELL-MANN and K. M. WATSON: *Annual Review of Nuclear Science*, **4**, 219 (1954); K. M. WATSON *et al.* ref. (24).

(+) It may be noticed that the approximations involved in the calculations of the curves presented by HAGERMAN *et al.* tend to exaggerate appreciably the actual difference between the two extreme cases $|L|^2/|K|^2 = 0$ and $|K|^2/|L|^2 = 0$.

(×) See ref. (17). The Penner points have been averaged over intervals wider than those used by the author.

Neglecting the Compton effect and other processes very improbable at the energies of the experiment referred to here, the interaction of a photon with deuterium can only produce either the photodisintegration or the emission of a single pion. In the free nucleons, of course, only the last process is possible.

With these approximations, if one assumes that the mode of absorption of a photon by the nucleons is independent from the final states, the following sum rule must be valid ([22]):

$$\sigma_p^+ + \sigma_n^- + \sigma_n^0 + \sigma_p^0 = \sigma_D^+ + \sigma_D^- + \sigma_D^0 + \sigma^{ph} , \tag{26}$$

where $\sigma_{n,p,D}^{\pm 0}$ is the total cross-section for photoproduction of $\pi^{\pm 0}$ from neutron, proton and deuterium, and σ^{ph} is the total cross-section for photodisintegration of deuterium.

Instead of (26) one may write:

$$\sigma^{ph} = (1+R)(1-A)\sigma_p^+ + (1-A_0)(\sigma_n^0 + \sigma_p^0) , \tag{27}$$

where

$$R = \frac{\sigma_D^-}{\sigma_D^+} = \frac{\sigma_n^-}{\sigma_p^+} , \qquad A = \frac{\sigma_D^+}{\sigma_p^+} ,$$

and $A_0 = (\sigma_D^0/(\sigma_n^0 + \sigma_p^0))$ is the attenuation factor for photoproduction of π^0 from deuterium with respect to the free nucleus. From (27) one can deduce that it must be:

$$\sigma^{ph} \geqslant (1+R)(1-A)\sigma_p^+ . \tag{28}$$

Assuming the value given in Table III of I for σ_p^+ at 200 MeV, and evaluating R and A from the data given in Tables VI, VII and VIII of this paper, one obtains $(43 \pm 6)\cdot 10^{-30}$ cm^2 for the second term in (28). At the same energy, the value for σ^{ph} is $(55 \pm 3)\cdot 10^{-30}$ cm^2 ([23]). The second term of (24) is not far from the upper limit because of the smallness of the cross-section for photoproduction of π^0 by free nucleons.

Actually, at 200 MeV ([24]) the cross-section for photoproduction of π^0 is not more than the 20% of the cross-section for π^+.

It is rather obvious that far from the threshold where the other causes

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([22]) D. CARLSON-LEE, G. STOPPINI and L. TAU: *Nuovo Cimento*, **2**, 162 (1955).

([23]) E. A. WHALIN, B. D. SCHRIEVER and A. O. HANSON: *Phys. Rev.*, **101**, 377 (1956).

([24]) K. M. WATSON, J. C. KECK, A. V. TOLLESTRUP and R. L. WALKER: *Phys. Rev.*, **101**, 1159 (1956); L. J. KOESTER, Jr. and F. E. MILLS: *Phys. Rev.*, **105**, 1900 (1957).

TABLE VIII.

$E_{\gamma\,\mathrm{lab}}$ \ ϑ_{lab}	$(d\sigma^-/d\sigma^+)_{45°}$	$(d\sigma^-/d\sigma^+)_{75°}$	$(d\sigma^-/d\sigma^+)_{103°}$	$(d\sigma^-/d\sigma^+)_{150°}$
170	1.67 ± .19	1.50 ± .05	—	—
180	1.35 ± .12	1.41 ± .10	1.47 ± .11	—
190	1.19 ± .10	1.41 ± .09	1.23 ± .16	1.38 ± .13
200	1.25 ± .12	1.28 ± .09	1.14 ± .15	1.46 ± .12
210	1.16 ± .12	1.05 ± .08	1.62 ± .21	1.37 ± .12
220	1.10 ± .12	1.26 ± .09	1.34 ± .16	1.31 ± .10
230	.96 ± .21	1.07 ± .08	1.32 ± .10	1.67 ± .13

of attenuation previously discussed (Pauli exclusion and kinematics) are gradually decreasing, reaction (9) remains the dominant fact which determines the difference between $d\sigma_H^+$ and $d\sigma_D^-$.

6. – The π^-/π^+ ratio.

6·1. – According to (19) the π^-/π^+ ratio in deuterium can be written

$$(29) \qquad R(\nu,\vartheta)=\frac{(d\sigma^-)_D}{(d\sigma^+)_D}\simeq\frac{\int_{\nu_t}^{\nu_{\max}}d\nu\int_{\omega}^{\omega+\Delta\omega}\frac{\eta\omega}{\nu}(H_1^-I_1-H_2^-I_2)\,d\omega}{\int_{\nu_t}^{\nu_{\max}}d\nu\int_{\omega}^{\omega+\Delta\omega}\frac{\eta\omega}{\nu}(H_1^+I_1-H_2^-I_2)\,d\omega}.$$

Table VIII gives the values of $R(\nu,\vartheta)$ obtained by our data.

It can be shown (see Appendix) that apart from Coulomb corrections $R(\nu,\vartheta)$ would approach rather closely the value of $R(\nu,\vartheta)_0$ for free nucleons.

Actually only the complexity of the problem of the Coulomb corrections has been the essential cause of the many doubts about the possibility of reaching a proper estimate of $R(\nu,\vartheta)_0$ via the experimental value of $R(\nu,\vartheta)$. These doubts have a quite different weight according to the different aims placed on the estimate of $R(\nu,\vartheta)_0$.

As a matter of fact, any speculation of this type tries to establish a rather arbitrary distinction among the several charge effects which creates a difference between the two-nucleon states, the proton and the neutron.

Particularly one has in mind the neutron-proton mass difference which automatically gives an infinite value of the π^-/π^+ ratio at the π^+ threshold, and the influence of the Coulomb interaction which again makes infinite the value of the π^-/π^+ ratio when both pion velocities are vanishing.

This situation makes the meaning of the threshold value of the π^-/π^+ ratio somewhat ambiguous and brings up the question of defining a lower limit of the incident photon energy at which the comparison between theory and experiment can be made without ambiguity.

A correct, though not too clear condition can obviously be established assuming for this lower limit a value such that at least the kinematical differences due to the n-p mass differences are negligible. It can be easily shown that this is true in our case. It remains that even for free nucleons any extrapolation from the good energy intervals to the threshold of the π^-/π^+ ratio is essentially an attempt to determine a parameter which should be introduced by the theory *a priori*. The threshold value of the π^-/π^+ ratio is not an experimental number that one may measure with better and better experiments. The theory itself should consistently indicate the connection between the threshold values and the value of the π^-/π^+ ratio really accessible by the experiment.

Going from the free nucleon reactions to reaction (3) the situation becomes immediately much more complicated because it involves simultaneously a three-body problem and the Coulomb corrections. The impulse approximation tries to overcome the first source of complication. As far as the second is concerned, it is well known that there are two opposite Coulomb corrections to take into account when one wants to derive the value of $R(\nu, \vartheta)_0$ from the experimental value of $R(\nu, \vartheta)$. The first is the Coulomb repulsion between the two final protons in reaction (3*a*) which depresses $\mathrm{d}\sigma_{\mathrm{D}}^-$ with respect to $\mathrm{d}\sigma_{\mathrm{D}}^+$; the second is the close analogue of the factor (frequently called the Gamow factor) introduced by FERMI ([25]) in the theory of β-decay. This second correction arises from the fact that the negative pion is originated in the positive field of the recoiling proton (originally a neutron) and of the other (spectator) proton. This last correction, of course, tends to emphasize reaction (3*a*) with respect to reaction (3*b*) when all other conditions (apart from the neutron-proton mass difference) are the same.

Probably the most complete and accurate work on the estimate of these corrections is that done by BALDIN ([26]) quite recently. The work of BALDIN represents a substantial improvement over the previous ones on this matter mainly because i) the influence of the nuclear forces between the two final

([25]) E. FERMI: *Zeits. für Phys.*, **88**, 116 (1934).
([26]) A. M. BALDIN: *Nuovo Cimento*, **8**, 569 (1958).

nucleons is properly considered, and ii) all Coulomb effects are consistently introduced via a perturbation term in the interaction Hamiltonian. The intuitive and clear result of the BALDIN calculation is that the first correction (BALDIN calls it the « negative » one) persists also at fairly high pion energies because the recoiling proton moves rather slowly; while the second (« positive ») correction rises up very steeply at photon energies below 170 MeV.

6·2. – To achieve the most direct comparison of our experimental data with the theoretical prediction we proceeded as follows.

Firstly, we evaluated numerically the theoretical cross-sections of reactions (1) and (2) as given by the paper of CHEW *et al.* ([27]). The matrix elements given by these authors are supposed to be correct for all terms up to first order in μ/M. The calculations have been performed at the several C.M. angles of interest to us. The ratio of these theoretical cross-sections as a function of energy gives the theoretical behaviour of the π^-/π^+ ratio (*).

In order to be compared with the experimental results, these « theoretical » ratios should be modified according to the correction suggested by BALDIN. This was done, applying the condition

$$|\boldsymbol{K}| = \left|\frac{\boldsymbol{K}_1 - \boldsymbol{K}_2}{2}\right| = \left|\frac{\boldsymbol{D}}{2}\right| = \left|\frac{\boldsymbol{K}_1 + \boldsymbol{K}_2}{2}\right|,$$

which seems to be a fairly good way of averaging the final nucleon momenta. The over-all procedure seemed to be rather appropriate, at least as far as $|L^{\pm}|^2/|K^{\pm}|^2 \ll 1$, *i.e.* in our specific case.

The comparison between the « theoretical » corrected π^-/π^+ ratios and the experimental data is presented in Figs. 4, 5 and 6. In Figs. 4 and 5 the two extreme angles of 60° and 160° (c.m.) are considered. The Baldin corrections have been applied to the experimental values. The solid lines are the curves evaluated according to reference ([27]).

In Fig. 6, which refers to the $\vartheta = 90°$ (c.m.), the experimental points (inclusive of some Cal. Tech. values) are uncorrected. Instead, the Baldin

([27]) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1345 (1957).

(*) The effective numerical evaluations of the cross-sections, starting from the quite complicated amplitudes given by CHEW *et al.*, have been worked out by one of us (M. BENEVENTANO) and then checked on many points by the other authors of this paper. Some checks have also been done comparing these calculations with some of those accomplished by E. GOLDWASSER. In spite of the fact that we do not exclude the occurrence of some mistake and some wrong sign in this very long and tedious work, we believe that the calculated curves represent the predictions of the dispersion relations without substantial errors.

correction was applied to the curve predicted by dispersion relations. The $\vartheta = 105°$ (c.m.) values show a similar trend and are in quite good agreement with some data at $\vartheta = 120°$ kindly communicated to us by Dr. DUDZIAK.

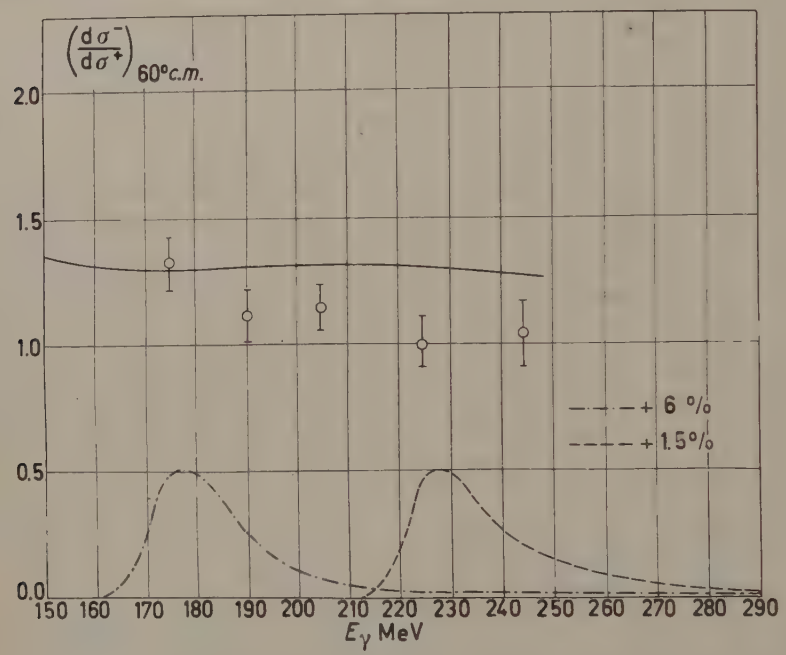

Fig. 4.

In Figs. 4 and 5 some « efficiency » curves are shown at the bottom of the figures. They have been itemized to show their dependence of E_γ and ϑ in the most extreme cases. Making use of the efficiency curves, in establishing the correlation among pion and photon energies the limits of the errors due to the two-body kinematics have been estimated. In Figs. 4 and 5 these errors are indicated. The sign of the errors is referred to the corresponding value of the π^-/π^+ ratio.

The disagreement may appear rather disappointing because it goes beyond the clearance which can be allowed for the most trivial experimental errors and the approximations claimed by the theoretical calculations. BALDIN's paper seems to be the most complete and accurate analysis made to date concerning the Coulomb corrections. At least in the energy region considered in our experiment and within the limits of the experimental errors, Baldin's calculations should be adequate.

On the other hand, the CHEW *et al.* paper is supposed to provide the correct amplitudes of free nucleons up to terms in μ/M. The other approximations

introduced for practical purposes in this paper again seem to be consistent with the energy interval to which our experimental results were extended.

The beam calibration was revised from the experimental side (*). From the point of view of the scanning efficiency the lowest energy point at 160° might well be affected by a fairly large (negative) error, but not larger than the statistical one. Scattering and absorption corrections were also reconsidered taking into account the somewhat different behaviour of negative pions as compared with positive pions.

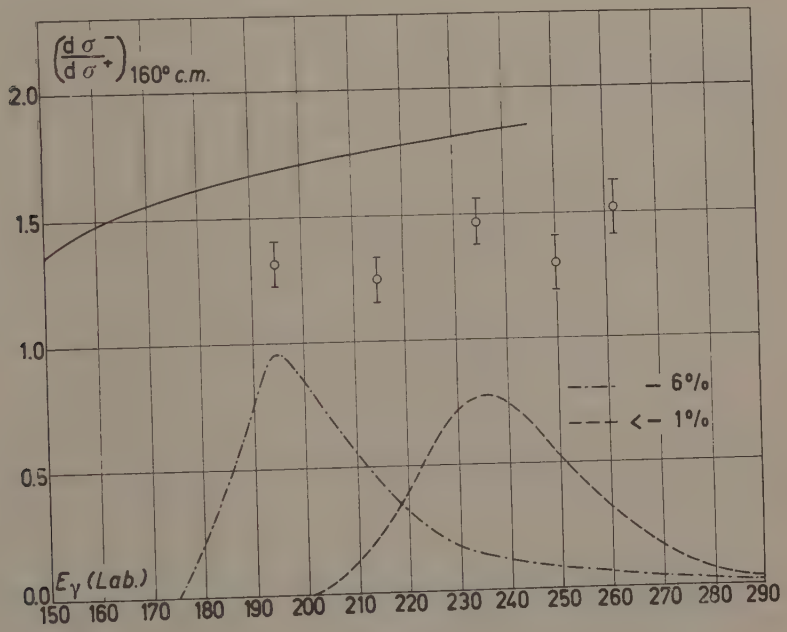

Fig. 5.

It was found that as far as the experimental data was concerned, the situation had to remain practically unchanged. Regarding the analysis of the data, we considered once again that its base, *i.e.* the use of the two-body kinematics, may have some very weak point. However, as has already been pointed out, the analysis is consistent with the scarce but direct information obtained with the pions observed in the high energy tail and with the results of the numerical evaluations which show that the contribution due to the high

(*) The authors are very indebted to Dr. C. Robinson, Physics Department, University of Illinois, who notified them of some possible misunderstanding in the meaning of the « standard conditions » of the monitor used by them during the performance of the experiment.

nucleon momenta does not appreciably affect the final figures of the π^-/π^+ ratio. In other words, it was found that only the use of some very peculiar deuteron w.f. would raise the high momenta contribution up to the point of making a bad approximation of the two-body kinematics. We further considered that the good agreement between the Cal. Tech. data (obtained with a maximum photon energy $E_{max} = 500$ MeV) and our experiment suggests an insensitiveness of results with respect to the photon spectrum which agrees with the main conclusion of the two-body kinematics.

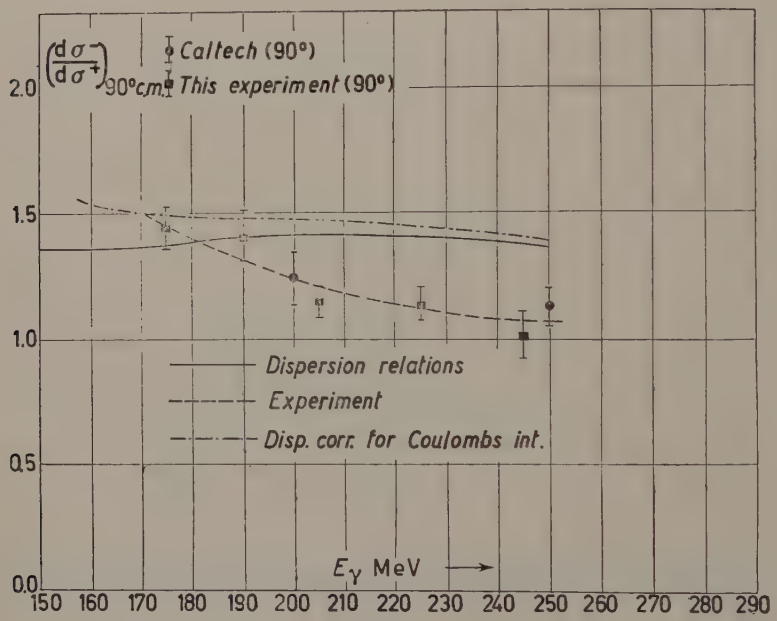

Fig. 6.

Finally, on the basis of the BALDIN work, we found that at very low energies where the two-body approximation is becoming worse and worse, our results agree quite well with the direct and complete experiment done on the π^- by ADAMOVIČ *et al.*

Concerning the influence of the Coulomb corrections, we may also add a somewhat trivial remark. Of the two opposite effects mentioned in the first paragraph of this section, the « *negative* » one should be dominant and spread over a fairly wide energy range at forward pion angles (45°). Instead, this negative effect has to be quite small when the pion is emitted at backward angles and the nucleon recoils in the forward direction with an average velocity with respect to the spectator nucleon much higher than in the first case. The

other correction (the positive one) depends mainly on the relative motion of the pion with respect to the father nucleon and depends very little upon the angle. Consequently, the negative effect which depresses the π^- with respect to the π^+ does not help in raising the experimental points at 160°; while it imposes a positive correction in the forward angles.

7. – Final remarks and conclusions.

As we have mentioned already, the experimental results presented in this paper have for a long time been considered somewhat puzzling, unsatisfactory and unclear. But they were not considered as any serious point of contradiction of the validity of the dispersion relation approach because of the unknown extent of the approximation involved, both in the analysis of the data and in the deduction of the formulae used in the CHEW *et al.* paper ([27]). In some respects one may even point out that apart from a common trend shown at all angles by the π^-/π^+ ratio (which is due to a common energy dependence) the dependence on ϑ does not contradict the prediction of the nucleon recoils associated with π-waves with $l \geqslant 1$. Actually the angular asymmetry in the π^-/π^+ ratio increases with the (average) incident photon energy as expected with a $\pm 20\%$ accuracy.

The difficulties, if any, are associated with the S-wave recoils (*). Recently, two attempts have been made to eliminate this unpleasant situation. It seems that ADAMOVIČ *et al.* ([28]) have been able to obtain from experimental data on the process (3*a*) values of the square of the matrix element for free neutron $|K^-|^2$ at four incident photon energies in the interval $\sim(153 \div 180)$ MeV. Comparing the results of an *empirical* extrapolation to threshold of the $|K^-|^2$ so obtained, with the corresponding empirical extrapolation reported in I (π^+ from *free* proton (+)), they found a π^-/π^+ ratio at threshold of 1.4 in agreement with the wanted value 1.3. Apart from relative calibration uncertainties, this good value remains, in this way, bound to an extrapolation to the threshold (that on the free proton), which leaves unsolved the difficulties connected with the Panofsky ratio already discussed many times ([29]). Instead,

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(*) It was already shown ([5]) that the S-wave recoil term R can be derived from the bulk of our results in a way which is certainly much less affected by the very low energy values than considering, for instance, the data at $\theta = 90°$ only.

([28]) M. I. ADAMOVIČ, G. V. KUZMIČEVA, V. G. LARIONOVA and S. P. HARLAMOV. Preprint kindly sent by authors.

(+) In the CERN Report of the 1958 Conference, p. 61, GOLDWASSER argued that the Russian authors used the Illinois *deuterium* data. Actually this was not the case.

([29]) See, for instance, CASSELS' report on the 1957 Rochester Conference and ref. ([28]).

starting from a pure speculative approach, the second attempt due to CINI *et al.* ([30]) proposed a brilliant solution of the threshold puzzle combining a Panofsky ratio $P = 1.5$, a π^-/π^+ ratio $R = 1.3$, and a $\omega = 1$ S-wave scattering length $(\alpha_3 - \alpha_1)/\eta = 0.24$. This tempting proposal (which will give a coupling constant closer to 0.07 than to 0.08) was discussed at the latest CERN Conference by PUPPI *et al.* ([31]).

Considering again only the photomesonic processes the proposal of CINI *et al.* implies a threshold value of $|K^-|^2$ at threshold quite higher than the one obtained by the empirical extrapolation. Some new data of BARBARO, CARLSON-LEE and GOLDWASSER ([31]) support this higher value which would be expected considering the influence of the so-called direct photoelectric term (see, for instance, ([31])). However, if one combines the value of $|K^-|^2$ with that of $|K^-|^2$ obtained by ADAMOVIČ *et al.* ([28]) at quite low energies (even lower than those involved in the experiment of BARBARO *et al.* ([31])) one finds for R a va lue less than one. For the sake of clarity and in order to be more explicit we would say that the situation seems to be the following. The work of ADAMOVIČ *et al.* (refs. ([20]) and ([28])) and the results of the Baldin calculations ([26]) *cannot* be considered consistent with the choice of the threshold parameters done by CINI *et al.* ([30]). In particular, those results cannot be considered a « justification » for that choice because the transition rate $\omega(\pi^- + p \to n + \gamma)$ derived from CINI *et al.* turns out to be about twice the value actually derivable from the papers of the Russian authors. Consequently, the agreement found in ref. ([30]), with the experimental value of the Panofsky ratio, appears *so far* to be based only on the extrapolation at threshold of the cross-section for positive pions produced by photons in hydrogen at 90° C.M. This extrapolation excludes the LEISS *et al.* experiment (see ref. ([30])) and « does not match well the apparent slope » of any set of experimental data between 170 and 230 MeV; but fortunately it seems to be in agreement with the preliminary results obtained at 160 MeV by BARBARO *et al.*

* * *

The authors are very grateful to Professors A. H. O. HANSON and E. GOLDWASSER for their assistance in performing the deuterium experiment; and they are deeply indebted to Professor L. LAVATELLI for his invaluable help in computing the functions I_1 and I_2. They also want to express their appreciation to DOROTHY CARLSON-LEE for her great contribution in the ana-

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([30]) M. CINI, R. GATTO, E. L. GOLDWASSER and M. A. RUDERMAN: *Nuovo Cimento* **10**, 242 (1958).

([31]) G. PUPPI: CERN Report of the 1958 Conference, p. 49 and following.

lysis of the data and to Mr. MENDOLA and Mr. SANTACROCE for the great skilfulness and clear understanding of the problems connected with the scanning of the pellicles.

APPENDIX

To compare $R(\nu, \vartheta)$ with the value $R(\nu, \vartheta)_0$ of the ratio π^-/π^+ for free nucleons, neglecting the Coulomb corrections, one may proceed as follows (remembering (19) and (25)) and, with the use of the mean theorem one may write

$$R(\nu, \vartheta) = R(\nu, \vartheta)_0 \frac{\Gamma^-}{\Gamma^+}, \tag{30}$$

where

$$R(\nu, \vartheta)_0 = \frac{\mathrm{d}\sigma_N^-}{\mathrm{d}\sigma_p^+}, \tag{31}$$

the Γ's are

$$\Gamma^\mp = 1 - \left[\left(\frac{1}{3} + \frac{|L_0^\mp|^2}{|K_0^\mp|^2}\right) \Big/ \left(1 - \frac{|L_0^\pm|^2}{|K_0^\mp|^2}\right)\right] \frac{F_2}{F_1}, \tag{32}$$

with

$$F_{1,2} = \int_{\nu_t}^{\nu_{\max}} I_{1,2} \frac{\mathrm{d}\nu}{\nu}.$$

Obviously $R(\nu, \vartheta)$ would be a good approximation of $R(\nu, \vartheta)_0$ if

$$\frac{|L_0^-|^2}{|K_0^-|^2} \lesssim \frac{|L_0^+|^2}{|K_0^+|^2}, \tag{33}$$

which is not true (see refs. ([2,4,5])). However, below 240 MeV the non-spin-flipping P-wave is always very small, as already seen in the discussion of the attenuation. Further $|K_0^-|^2 > |K_0^+|^2$ because $\mathrm{d}\sigma_n > \mathrm{d}\sigma_p$, and this is due, at low energies, to the contribution of the nucleon recoils to the S-wave. Thus both $|L_0^-|^2/|K_0^-|^2$ and $|L_0^+|^2/|K_0^+|^2$ are very small and if any

$$\frac{|L_0^-|^2}{|K_0^-|^2} = \frac{|L_0^+|^2}{|K_0^+|^2}.$$

To estimate more properly the deviation of $R(\nu, \vartheta)$ from $R(\nu, \vartheta)_0$ we can apply a method of successive approximation.

The product $\mathrm{d}\sigma_p \times R(\nu, \vartheta)$ is supposed to be the zero approximation of $\mathrm{d}\sigma_N$. Then neglecting, for the reasons already pointed out in I, the influence of the

retardation term, we assume it appropriate to write (with the same notations used in I)

$$d\sigma_p \times R(\nu, \vartheta) \simeq d\sigma_N = W(a_0^- + a_1^- \cos\vartheta + a_2^- \cos^2\vartheta) .$$

Averaging in the energy interval (180÷220) MeV, we found

$$a_1^-/a_1^+ = 2.1 \pm .2; \qquad a_2^-/a_2^+ = .9 \pm .4 ,$$

and for a_0^- the following values

ν	$170 \div 190$	$200 \div 220$,
a_0^-	$2.12 \pm .12$	$1.76 \pm .10$.

From these values, making use of the well-known multipole analysis (see, for instance, I), a first approximation for $|L^-|^2/|K^-|^2$ can be derived. The approximation can then be improved by interaction. The conclusion wolud be that the deviations of $R(\nu, \vartheta)$ from $R(\nu, \vartheta)_0$ could not be larger than 3% because (33) does not hold.

A similar conclusion would be reached if one wolud consider the influence of the *nuclear* forces alone in the final two-nucleons state. Apart from the *Coulomb* interaction they are supposed to be the same in the reactions (3*a*) and (3*b*). They tend to emphasize both reactions as they do, for instance, in the photodisintegration. Hence, *the only appreciable deviations of* $R(\nu, \vartheta)$ *from* $R(\nu, \vartheta)_0$ *are due to the Coulomb forces.*

RIASSUNTO

Viene presentata un'analisi sperimentale del processo

$$h\nu + D \nearrow \pi^- + p + p \quad \searrow \pi^+ + n + n$$

Il rapporto $\sigma(-)/\sigma(+)$ è stato misurato nell'intervallo di energia dei fotoni compreso tra 170 e 230 MeV. I risultati sono stati esaminati usando l'« impulse approximation » con lo scopo di ottenere informazioni sul processo $h\nu + n \to \pi^- + p$.

Energy Dependence of the Asymmetry Parameter of π-μ-e Decay in Nuclear Emulsions.

C. Besson and V. Brisson-Fouché

Ecole Polytechnique, Laboratoire de Physique - Paris

(ricevuto l'8 Settembre 1958)

915 π^+-μ-e at rest have been investigated in order to study the energy dependence of the asymmetry in the angular distribution of the electrons.

The 1000 μm thick emulsions have been exposed to the π^+ beam of the Chicago synchrocyclotron. The plates were shielded from external magnetic field. In order to avoid systematic errors in the angular distribution, they were scanned only for the π-μ, and not for the μ-e.

The following criteria have been used in selecting the events:

1) The whole π-μ-e decay lies in the same plate.

2) The μ-meson stops at more than 50 μm from the surfaces of the emulsion.

3) The length of the positron track is at least 1600 μm, and its dip at the start is less than 15%. From these restrictions, the fraction of slow electrons lost because of large scattering angles is about 10% at 5 MeV and 3% at 10 MeV.

The energies of the electrons are derived from scattering measurements, with elimination between 25 μm and 50 μm-cells, using a constant total length of 1600 μm, which corresponds to a statistical error of 15%.

Measurements and calculations have been made by an automatic apparatus [1] However, the energies <15 MeV have been only estimated, not measured.

The spatial angle between the momenta $\boldsymbol{p}_e$ and $\boldsymbol{p}_\mu$ of the electron and muon at emission has been measured for each event, with an accuracy of about 1%.

1. – Results.

1·1. *Asymmetry.* – For an angular distribution $dN/d\Omega = 1 - a\cos\theta$, the average value of $\cos\theta$ is $a/3$, or $a = 3\overline{\cos\theta}$, the error being $\sigma_a = \sqrt{3/N}$. In order to evaluate the energy dependence of a, the events have been grouped in energy intervals of 10 MeV, except for the low energies (<15 MeV). The experimental values of a for each interval are pre-

[1] V. Brisson-Fouché: *Compt. Rend. Acad. Sci.*, **245**, 1800 (1958).

sented in Table I. These values are compared in Fig. 1 with the calculated distribution deduced from the « two-components neutrino theory »,

$$a = -(1-2E)/(3-2E)\,,$$

assuming a 50% depolarization of the muon in emulsion (curve *a*). The curve *b*

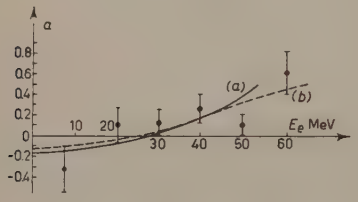

Fig. 1.

is deduced from the latter by correcting for the statistical dispersion in the scattering measurements (about 15%), and bremsstrahlung losses (about 2%).

TABLE I.

Energy (MeV)	Number	$a = 3\overline{\cos\theta}$
0÷15	74 (*)	-0.301 ± 0.201
15÷25	100	$+0.113 \pm 0.173$
25÷35	199	$+0.117 \pm 0.122$
35÷45	224	$+0.278 \pm 0.115$
45÷55	234	$+0.095 \pm 0.113$
55	83	$+0.606 \pm 0.190$
Total	914	$a = 0.183 \pm 0.058$

(*) Including 14 events from a special search of low energy electrons.

The experimental results agree with this calculated curve, and with those obtained in similar experiments ([2,3]).

The mean value of *a*, integrated over the whole energy spectrum, is

$$a = 0.171 \pm 0.058\,.$$

This does not include bremsstrahlung and scattering dispersion corrections. However, we have calculated the geometrical correction due to the biased selection of the events (all steep electrons and many steep muons are rejected). The corrected value is:

$$a = 0.183 \pm 0.058\,,$$

corresponding to a depolarization of $(57 \pm 17)\%$ (*).

1·2. *Energy spectrum – ϱ-value.* – The experimental energy spectrum of the positrons is shown in Fig. 2. Statistical dispersion in the scattering measurements and bremsstrahlung losses are included in the calculation of the Michel parameter ϱ. Bremsstrahlung losses are quite small (about 2%) because the measured length of 1600 μm corresponds only to $5 \cdot 10^{-2}$ radiation length.

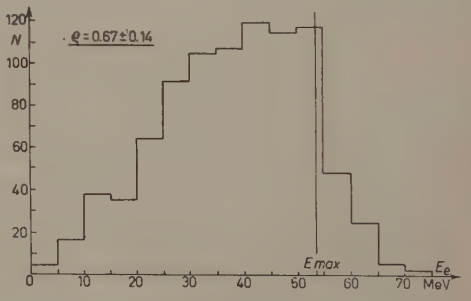

Fig. 2.

ϱ has been calculated from the second order momentum ([4]). For the scattering constant, the value of Pickup and Voyvodic has been taken ([5]). The result is:

$$\varrho = 0.67 \pm 0.14\,.$$

([2]) C. CASTAGNOLI: *International Conference*, (Padua-Venice, Sept. 1957).

([3]) V. A. SMIRNITSKY and A. O. WEISSENBERG: *Nucl. Phys.*, **5**, 33 (1958).

(*) The values in each energy interval were not corrected (Table I, Fig. 1), this correction being small compared to statistical errors.

([4]) H. BRAMSON *et al.*: *Phys. Rev.*, **88**, 304 (1952).

([5]) E. PICKUP and L. VOYVODIC: *Phys. Rev.*, **81**, 890 (1951).

However, no correction has been made for the inner bremsstrahlung; this correction would increase it by about 0.04 (6).

The error on ϱ is given by:

$$\sigma_\varrho^2 = (\sigma_\varrho)^2_{\text{statis.}} + (\sigma_\varrho)^2_{\text{brems.}} + (\sigma_\varrho)^2_{m_\mu} + (\sigma_\varrho)^2_K ,$$

where $(\sigma_\varrho)^2_{m_\mu}$ is the error due to the uncertainty on the μ-mass and $(\sigma_\varrho)^2_K$ is the error due to the uncertainty on the scattering constant K.

a) $(\sigma_\varrho)^2$ statistical, (900 events) is $27.5 \cdot 10^{-4}$.

b) $(\sigma_\varrho)^2_{\text{brems.}}$, due to the fluctuations of radiation losses, is $14.5 \cdot 10^{-4}$.

c) The present best value of m_μ being: $m_\mu = (206.86 \pm 0.11)\, m_e$, $(\sigma_\varrho)^2_{m_\mu} = 3.5 \cdot 10^{-4}$.

d) $(\sigma_\varrho)^2_K$ is $144 \cdot 10^{-4}$. (One can easily see that for a ϱ-value of 0.67, $\Delta\varrho/\varrho \neq 9\,\Delta K/K$. We have taken $\Delta K/K < 2\%$ (7)).

Finally, we have:

$$(\sigma_\varrho)^2 = (27.5 \cdot 10^{-4}) + (14.5 \cdot 10^{-4}) + (3.5 \cdot 10^{-4}) + (144 \cdot 10^{-4}) = 189 \cdot 10^{-4} ,$$

and

$$\sigma_\varrho = 0.14 .$$

* * *

We are greatly indebted to Professor TELEGDI for having supplied us with the exposed and processed emulsions.

(6) J. KINSHITA and A. SIRLIN: *Phys. Rev.*, **107**, 638 (1957).

(7) A. BONETTI, R. LEVI SETTI, M. PANETTI and G. TOMASINI: *Nuovo Cimento*, **3**, 33 (1956).

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

On the Symmetry of Elementary Particles (*).

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(ricevuto il 3 Ottobre 1958)

In this note we indicate that the symmetry expressed in the Gell-Mann-Nishijima scheme of elementary particles [1] may be conveniently described geometrically by an hexagonal structure (Fig. 1).

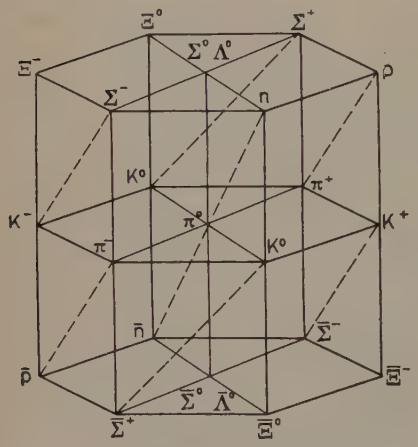

Fig. 1.

The three horizontal planes are planes of N (baryon number) = const and at the same time s (spin) = const. The three vertical planes (one along the Σ^+p-line and two others parallel to it) are planes of Q (electric charge) = const. The three vertical planes (one along the pn-line and two others parallel to it) are planes of $Y = N + S$ (hyperon charge) = const. Planes of S (strangeness) = const are indicated by dotted lines. The lower half of the diagram contains antibaryons which are obtained from the baryons by reflection through π^0. Using the conservation of N, Q, S or ΔS it is easy to read off from the diagram all possible production and decay reactions.

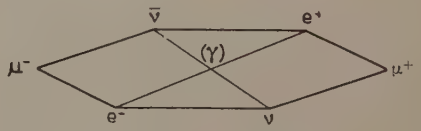

Fig. 2.

We remark that the leptons may also be placed in a somewhat degenerate hexagonal structure (Fig. 2), their electric charge planes agree with that of other

(*) Supported in part by the Air Force Office of Scientific Research.

(1) M. Gell-Mann: *Phys. Rev.*, **92**, 883 (1953); J. Schwinger: *Phys. Rev.*, **104**, 1164 (1956); M. Kretzschmar: *Nuovo Cimento*, **8**, 949 (1958).

particles. Antiparticle reflection will be affected here through the center as for the bosons.

All particles known definitely at present fit in the diagram. If there is anything fundamental about this *hexagonal symmetry* than the only possible way of having new particles would be a doubling of positions; for example, another neutral $T=0$ meson at the place of π^0 in analogy to Λ^0 [2].

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[2] A. M. BALDIN and P. K. KABIR: *Nuovo Cimento*, **9**, 547 (1958).

IL NUOVO CIMENTO Vol. X, N. 6 16 Dicembre 1958

Bubble Chamber Detector of Weak Radioactivity.

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(ricevuto il 7 novembre 1958)

The bubble chamber is an instrument widely used in high energy physics to visualize the tracks of ionizing particles. A clean bubble chamber, however, can be used as a detector of very weak radioactivity. In this type of bubble chamber, boiling is usually due to ionizing events; a simple recording of the boiling frequency gives a measurement of the ionizing radiation intensity.

Thus, calling t the average sensitive time, $n = 1/t$ gives the number per s of the ionizing particles going through the chamber and capable of starting the boiling (after background is subtracted).

The chamber is a pyrex bulb of about 5 cc, the moving diaphragm is in mylar backed by teflon. An expansion ratio control, a safety device against extra-pressures and a mechanical detector of boiling are provided. The latter controls an electronic chronometer. The filling liquid usually was diethyl ether.

We report here some preliminary measurements giving indications on the sensitivity to protons, β and α particles of the apparatus.

Fig. 1, shows $n = 1/t$ plotted *vs.* temperature, for the various types of radiation in curves I, II, IV after subtraction of background (curve III).

In order to have electrons stopping in the liquid we have used an external 1 millicurie γ-ray source (^{137}Ca) placed at 1.65 m; curve IV (*) is obtained with the Cs source. Curve II shows the sensitivity of the chamber to a neutron flux from a 10 neutron/s RaBe source placed at a distance of 100 cm (γ-rays were shielded by a 10 cm lead absorber). The plateau between 120 °C and 133 °C presumably corresponds to a 100% efficiency in recording the recoil protons. Curve I gives the results obtained after solution of 10 mg/l of uranyl nitrate in the ether used to fill the chamber. The plateau corresponding to a 100% efficiency for α-particles is shifted to lower temperatures with regard to thatobtained for protons. The rise at about 100 °C, presumably of different

(*) Note the change of scale and shift of origin in the ordinates of curve IV only.

origin, is now under study.

Up to 115° the background is negligible. The average sensitive time of our chamber filled with pure diethyl ether (without sources of radiation) is of the order of hours for temperatures corresponding to the plateau of α-particles and of tens of minutes for temperatures corresponding to the proton plateau. Moreover the background turns out to be practically constant in successive fillings of the chamber.

If we consider the characteristics of the sources used and the corresponding observed intensities, we can conclude that, due to the extremely low background our instrument has proved to be an high efficiency fast neutron monitor and an instrument also capable of detecting and measuring very weak α radiation with high counting stability.

Further systematic measurements are in progress.

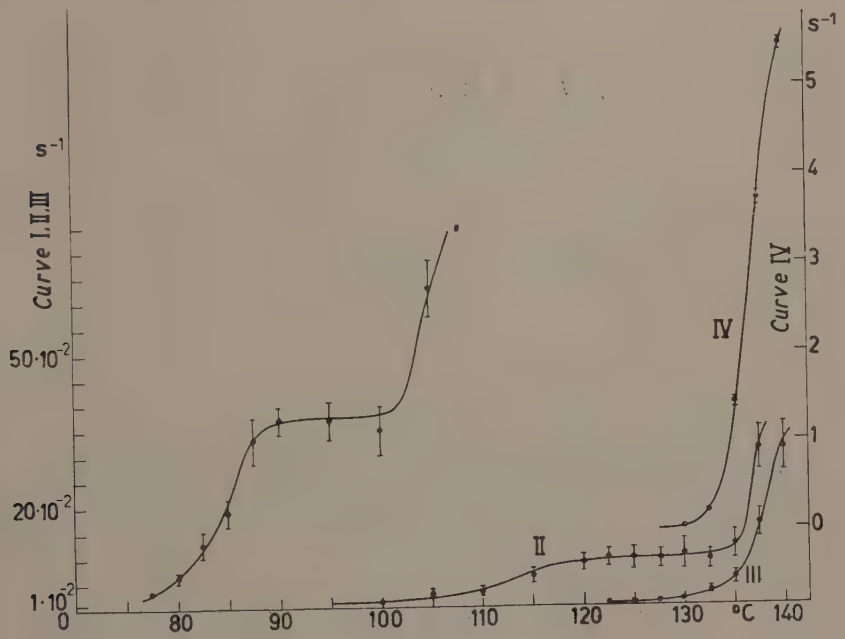

Fig. 1.

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The Field Theory with Yukawa Coupling in One Dimension.

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In a very recent paper [1] V. Glaser has found the solution of the equation of motion and the S matrix in the Thirring model [2,3]. The aim of this note is to show that one can solve exactly the equations of motion and find the S matrix also in the case of a zero rest mass spinor field interacting with a vector field. Depending on whether one considers a boson field with vanishing (electromagnetic field) or non-vanishing rest mass one has either

$$H_I = e\bar{\psi}\gamma_\mu\psi A^\mu , \tag{1}$$

or

$$H_I = g\bar{\psi}\gamma_\mu\psi A^\mu + g^2\bar{\psi}\psi\bar{\psi}\psi , \tag{2}$$

The contact term arises in the usual way [4].

In the following we shall deal with the electromagnetic field only. The case of the vector meson field can be examined in a very similar manner. The operator $U(\sigma, \sigma_0)$ which transforms the states in the interaction picture is defined usually by the equations

$$\begin{cases} i\dfrac{\delta}{\delta\sigma(x)}U(\sigma,\sigma_0) = H_I(x)\,U(\sigma,\sigma_0) , \\ -i\dfrac{\delta}{\delta\sigma_0(x)}U(\sigma,\sigma_0) = U(\sigma,\sigma_0)H_I(x) . \end{cases} \tag{3}$$

If we take into account the commutation rules for the current operator j_μ [1] and the commutation rules for the potentials

$$[A_\mu(x), A_\nu(x')] = \frac{1}{i}g_{\mu\nu}D(x-x') , \tag{4}$$

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[1] V. Glaser, *Nuovo Cimento* **9**, 990 (1958).
[2] W. Thirring *Nuovo Cimento* **9**, 1007 (1958).
[3] W. Thirring, *Ann. of Phys.* **9**, 91 (1958).
[4] H. Umezawa, *Quantum Field Theory* (Amsterdam, 1956).

we can obtain the solution of the equations (3) in the form

$$(5)\qquad U(\sigma,\sigma_0)=\exp\left[-ie\int_{\sigma_0}^{\sigma}j_\mu A^\mu\,\mathrm{d}^2x\right]\exp\left[-ie^2\int_{\sigma_0}^{\sigma}\int_{\sigma_0}^{\sigma}j_\mu(x)\,\bar{D}(x-x')j^\mu(x')\,\mathrm{d}^2x\,\mathrm{d}^2x'\right].$$

Here

$$g_{00}=1\,,\qquad g_{11}=-1\,,$$

$$(6)\qquad \begin{cases} D(x-x')=(2\pi)^{-2}\int_C \mathrm{d}^2k\,k^{-2}\exp\left[-ik_\mu x^\mu\right]=D_A-D_R\,,\\ \bar{D}(x-x')=(2\pi)^{-2}P\int \mathrm{d}^2k\,k^{-2}\exp\left[-ik_\mu x^\mu\right]=\dfrac{1}{2}(D_A+D_R)\,. \end{cases}$$

With the notations

$$j_+=\psi_1^*\psi_1\,,\qquad j_-=\psi_2^*\psi_2\,,\qquad A_\pm=A^0\pm A^1\,,\qquad x_\pm=x^0\pm x^1\,,$$

the formula (5) gives

$$(7)\qquad D(\sigma,\sigma_0)=\exp\left[-ie\int_{\sigma_0}^{\sigma}(j_+A_+ + j_-A_-)\,\mathrm{d}^2x\right]\exp\left[-2ie^2\int_{\sigma_0}^{\sigma}\int_{\sigma_0}^{\sigma}j_+(x)\,\bar{D}(x-x')j_-(x')\,\mathrm{d}^2x\,\mathrm{d}x'\right]$$

$$(8)\qquad \begin{cases} U(\sigma,\sigma_0)=\exp\left[-ie\int_{\sigma_0}^{\sigma}j_+A_+\mathrm{d}^2x\right]\exp\left[-ie\int_{\sigma_0}^{\sigma}j_-A_-\mathrm{d}^2x\right]\cdot\\ \qquad\qquad\cdot\exp\left[-2ie^2\int_{\sigma_0}^{\sigma}\int_{\sigma_0}^{\sigma}j_-(x)\,D_R(x-x')j_+(x')\,\mathrm{d}^2x\,\mathrm{d}^2x'\right],\\ U(\sigma,\sigma_0)=\exp\left[-ie\int_{\sigma_0}^{\sigma}j_-A_-\,\mathrm{d}^2x\right]\exp\left[-ie\int_{\sigma_0}^{\sigma}j_+A_+\,\mathrm{d}^2x\right]\cdot\\ \qquad\qquad\cdot\exp\left[-2ie^2\int_{\sigma_0}^{\sigma}\int_{\sigma_0}^{\sigma}j_+(x)\,D_R(x-x')j_-(x')\,\mathrm{d}^2x\,\mathrm{d}^2x'\right]. \end{cases}$$

The S matrix can be obtained from (5) by the usual limiting process

$$(9)\qquad S=\lim_{\sigma_0\to-\infty\ \sigma\to\infty}U(\sigma,\sigma_0)\,.$$

To obtain the solutions of the field equations in terms of the free field operators

it is sufficient to compute the expressions

$$U^{-1}(\sigma, \sigma_0)\, A_\mu(x)\, U(\sigma, \sigma_0) = \underset{-}{A}_\mu(x)\,, \tag{10}$$

$$U^{-1}(\sigma, \sigma_0)\ \ \psi(x)\, U(\sigma, \sigma_0) = \ \underset{-}{\psi}\,(x)\,. \tag{11}$$

This can be done for arbitrary x, σ, σ_0 and yields

$$\underset{-}{A}_\mu(x) = A_\mu(x) + e\int_{\sigma_0}^{\sigma} D(x - x')\, \mathrm{d}^2x' j_\mu(x')\,, \tag{12}$$

$$\begin{cases} \underset{-}{\psi}_1(x) = \exp\left[- ie\int_{\sigma_0}^{\sigma} \delta(x_+ - x'_+)\, \mathrm{d}^2x' \left(A_+(x') + 2e\int_{\sigma_0}^{\sigma} D_R(x' - x'') j_-(x'')\, \mathrm{d}^2x'' \right) \right] \cdot \psi_1(x)\,, \\ \underset{-}{\psi}_2(x) = \exp\left[- ie\int_{\sigma_0}^{\sigma} \delta(x_- - x'_-)\, \mathrm{d}^2x' \left(A_-(x') + 2e\int_{\sigma_0}^{\sigma} D_R(x' - x'') j_+(x'')\, \mathrm{d}^2x'' \right) \right] \cdot \psi_2(x)\,. \end{cases} \tag{13}$$

It is worth-while to notice that the assumption concerning the vanishing of the fermion rest mass is in this model, like in the Thirring model, the necessary condition to solve the problem in a simple manner. Further considerations on this model will be published later.

LIBRI RICEVUTI E RECENSIONI

L. D. Landau and E. M. Lifshitz – *Quantum Mechanics. Non relativistic Theory*. Pergamon Press, London-Paris, 1958, pp. 1+515.

Questo libro fa parte di un trattato generale di Fisica Teorica in nove volumi, a cui i due illustri fisici russi stanno lavorando da anni e che la Pergamon Press presenta ora al pubblico occidentale nella traduzione in lingua Inglese. Non si può quindi prescindere dal fatto che l'attuale volume non forma un tutto a sè ma è parte di un insieme più vasto, se si vuole capire la scelta degli argomenti trattati. Tenendo conto dei criteri che hanno ispirato tale scelta agli autori e cioè: *a*) il sottotitolo stesso *Non relativistic theory*, interpretato come essi spiegano nella prefazione, nel senso di scartare tutti i fenomeni dipendenti in modo significativo dalla velocità della luce, destinandoli ad un altro volume; *b*) l'inclusione nel trattato di quei soli argomenti che abbiano raggiunto un grado di stabilità e di sicurezza tale da consentire una trattazione indipendente dalla discussione o citazione di materiale sperimentale (come avviene nel caso della Fisica Nucleare, che, per tale ragione, non viene inclusa nel piano generale del trattato), si può dire l'opera in effetti ricopra il campo restante della meccanica quantistica che i criteri sopracitati hanno in tal modo delimitato.

L'originale volume in lingua russa apparso nel 1948 non è stato rimaneggiato in modo sostanziale nell'edizione inglese; dato però che gli argomenti trattati hanno assunto oramai una notevole consistenza e coerenza, una revisione od aggiornamento non si rende particolarmente necessario.

Come è ben noto, la meccanica quantistica si può insegnare generalmente seguendo due diverse vie: o partire dalla discussione concettuale delle condizioni di osservabilità del mondo atomico per fondarvi in modo intuitivo le relazioni di indeterminazione e le ipotesi sul significato fisico delle funzioni d'onda e svolgere di qui l'aspetto ondulatorio della meccanica quantistica opportunamente integrato dall'aspetto matriciale, il che costituisce il metodo storico-intuitivo seguito in genere dai trattati più antichi; oppure partire dall'impostazione matematica astratta e rigorosa delle proprietà degli stati e degli operatori nello spazio di Hilbert e dedurne le applicazioni fisiche quale conseguenza dei postulati esatti intuibili che stabiliscono la corrispondenza tra le entità matematiche e le realtà fisiche, come è stato tipicamente svolto nel libro di Dirac. Queste due impostazioni costituiscono come due stadi necessari per l'esposizione completa della meccanica quantistica, l'una essendo di per sè incompleta ed insufficiente, l'altra invece troppo astratta per poter essere pienamente capita senza previa preparazione. Sembra che sia difficile assumere una posizione intermedia capace di unire in un'unica esposizione

l'intuibilità al rigore, alla completezza ed all'ordine logico, ed i trattati che hanno tentato di farlo sono spesso risultati mancanti di qualche componente importante. Neppure questo volume è esente da tale difetto, dato che cerca appunto di alternare l'impostazione di certi principi (incertezza delle misure, significato delle funzioni d'onda) sulla base della discussione concettuale della esperienza alla deduzione del formalismo matematico in una versione alquanto semplificata, che tuttavia predomina e soffoca alquanto la parte intuitiva. Di conseguenza da un lato tale parte intuitiva ci sembra insufficiente per una adeguata comprensione del significato dei postulati basilari della meccanica quantistica, e tale insufficienza conduce a sconsigliare il libro come guida di un corso a se stante, a meno che il suo studio non venga integrato da una introduzione propedeutica sufficientemente ampia; d'altro lato la parte matematica formale, per volersi adeguare a quei pochi richiami alla base sperimentale, sembra spesso presentata in modo occasionale ed alquanto disordinato che sminuisce il senso della completezza e della coerenza dell'insieme del formalismo così importante per apprezzarne il vero valore.

Questo difetto iniziale (quasi esclusivamente concentrato nei due primi capitoli riguardanti l'impostazione generale vista principalmente nella rappresentazione dei momenti e dell'energia) è tuttavia compensato dai molti pregi delle varie parti comprendenti principalmente: la trattazione generale dell'equazione di Schrödinger colle proprietà dei momenti angolari e lo studio del moto in un campo di forze centrali (cap. 3, 4, 5); la teoria delle perturbazioni negli aspetti statico e dinamico ed i metodi semiclassici che portano al legame colla teoria di Bohr e la fisica classica (cap. 6 e 7); le proprietà degli spinori e l'incidenza dello spin nei problemi a più particelle identiche (cap. 8 e 9), quindi l'applicazione di queste proprietà generali a sistemi particolari: l'atomo (cap. 10) e le molecole (cap. 11 e 13), completate con i fondamenti della teoria delle simmetrie (cap. 12); ed infine la teoria dell'urto elastico ed anelastico (cap. 14 e 15) ed il moto in campo magnetico (cap. 16).

Tra questi pregi che sarebbe cosa troppo lunga elencare in dettaglio, possiamo notare lo studio particolareggiato delle soluzioni dei vari problemi in funzione delle condizioni fisiche, cosa che in molte opere viene spesso trascurata e che porta invece ad una comprensione più intima, sia del significato fisico della soluzione matematica e della sua dipendenza dai vari parametri, sia dell'adeguatezza delle approssimazioni adottate per tradurre la natura del problema e per risolverle; la formulazione concentrata ed unitaria delle proprietà di certe grandezze importanti o di certi complessi di leggi che vengono invece spesso sottintesi o presentati in modo sparso, come nel caso dei momenti angolari, degli spinori e delle simmetrie; e per finire, un tono generale di esposizione chiara ed obbiettiva che rende la lettura in genere facile ed attraente.

In complesso quindi lo studio dell'opera può risultare di grande utilità per chi voglia approfondire avendone già una conoscenza elementare, la sua comprensione effettiva ed applicativa e non puramente formale della meccanica quantistica. L'opera può inoltre servire di guida a chi volesse impostare o superare punti delicati in vari problemi che vengono spesso sorvolati.

N. Dallaporta